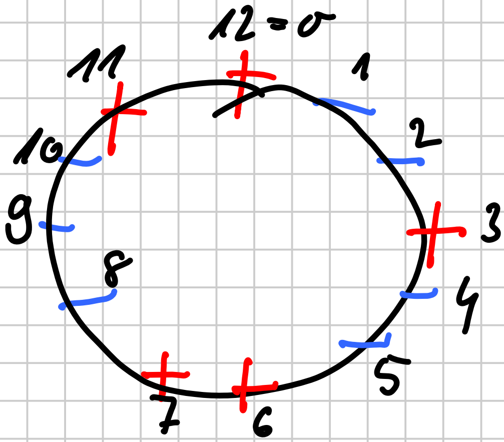
# Ising Model 1D

exactly solvable models:

- non-interacting (mostly trivial, no transitions)
- often in infinite dimensions or equivalently infinite range (next)
- → one dimension, spins on a chain




$$\mathcal{H}_N = -J\sum_{i=0}^{N-1} s_i s_{i+1} - H\sum_{i=0}^{N-1} s_i$$

- each neighboring pair counted only once
- periodic boundary conditions $s_N = s_0$

partition function $Z$
(reminder: If we can write down $Z$, we can calculate everything: we have solved the problem!)

$$Z = \sum_{s_0=\pm1}\sum_{s_1=\pm1}\cdots\sum_{s_{N-1}=\pm1} e^{-\beta\mathcal{H}}$$

$$= \sum_{s_0=\pm1}\sum_{s_1=\pm1}\cdots\sum_{s_{N-1}=\pm1} \exp\left\{\beta J\sum_{i=0}^{N-1} s_i s_{i+1} + \frac{\beta H}{2}\sum_{i=0}^{N-1} s_i + s_{i+1}\right\}$$

$$= \sum_{s_0=\pm 1} \sum_{s_1=\pm 1} \cdots \sum_{s_{N-1}=\pm 1} \prod_{i=0}^{N-1} \exp\left\{ \beta J s_i s_{i+1} + \beta \frac{H}{2} (s_i + s_{i+1}) \right\}$$

for given $i$, the following states of spin $s_i$ and $s_{i+1}$ are involved

| | $s_{i+1}=+1$ | $s_{i+1}=-1$ |
|---|---|---|
| $s_i=+1$ | $+1$ $\quad +1$ | $-1$ $\quad 0$ |
| $s_i=-1$ | $-1$ $\quad 0$ | $+1$ $\quad -1$ |

and $s_i s_{i+1}$
$\frac{1}{2}(s_i + s_{i+1})$

suggesting the following matrix

$$T_{i,i+1} = \exp\left\{ \beta J s_i s_{i+1} + \beta \frac{H}{2} (s_i + s_{i+1}) \right\}$$

$$T = \begin{pmatrix} e^{\beta J + \beta H} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J - \beta H} \end{pmatrix}$$

$$\mathcal{Z} = \sum_{s_0=\pm 1} \sum_{s_1=\pm 1} \cdots \sum_{s_{N-1}=\pm 1} T_{0,1} \cdot T_{1,2} \cdots T_{N-1,0} =$$

$$= \sum_{s_0=\pm 1} \left( T^N \right)_{0,0}$$ transfer matrix $T$

only diagonal elements remain

↳ trace of matrix $T^N$

↳ sum of its eigenvalues

$$\mathcal{Z} = \sum_i \lambda_i^N$$

after some straightforward calculations we get eigenvalues of $T$:

$$\lambda_\pm = e^{\beta J} \cosh \beta H \pm \sqrt{e^{2\beta J} \cosh^2 \beta H - 2 \sinh 2\beta J}$$

hence $\mathcal{Z} = \lambda_+^N + \lambda_-^N = \lambda_+^N \left(1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right) \approx \lambda_+^N$

as $(\lambda_- / \lambda_+)^N \to 0$ for large $N$

in the end: $\boxed{\mathcal{Z} = \lambda_+^N}$ in the thermodynamic limit $N \to \infty$

i.e. only the largest eigenvalue of the transfer matrix needs to be known, not the full spectrum of the matrix

NB dimension of transfer matrix in 1D

$+ + - + + + -$ 1st neighbor, 2x2 matrix

$+ + - + + + -$ 2nd neighbor, 4x4 matrix or for more than 2 states

+ − + 0 0 − 3 states : 3×3 matrix

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free energy per spin for large N

$$\frac{1}{N}F = -k_B T \lim_{N\to\infty} \frac{1}{N} \ln Z_N = -k_B T \ln \lambda_+$$

$$= -k_B T \ln\left\{ e^{\beta J} \cosh \beta H + \sqrt{e^{2\beta J} \cosh^2 \beta H - 2 \sinh 2\beta J} \right\}$$

↳ symmetric for $H \to -H$

magnetization per spin

$$m = -\partial_H \frac{F}{N}\Big|_{H=0} = k_B T \, \partial_H \ln \lambda_+ \Big|_{H=0} =$$

$$= \frac{k_B T}{\lambda_+} \partial_H \lambda_+ \Big|_{H=0} = 0$$

$\boxed{\text{Hence, no phase transition for Ising model in 1D.}}$

# Ising Model for infinite range or dimension

$$\mathcal{H} = \frac{-J}{2N}\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N} s_i s_j - H\sum_{i=1}^{N} s_i$$

again

$$\mathcal{Z} = \sum_{s_1=\pm1}\cdots\sum_{s_N=\pm1} e^{-\beta\mathcal{H}} = \sum_{s_1=\pm1}\cdots\sum_{s_N=\pm1} \exp\left\{ \frac{\beta J}{2N}\left[\sum_{i=1}^{N}\sum_{j=1}^{N} s_i s_j - \sum_{i=1}^{N} s_i^2\right] + \beta H \sum_{i=1}^{N} s_i \right\}$$

now $\sum_{i=1}^{N} s_i^2 = N\,(\pm 1)^2 = N$

define $x := \sum_{i=1}^{N} s_i$

$$\sum_{i=1}^{N}\sum_{j=1}^{N} s_i s_j = \sum_{i=1}^{N} s_i \sum_{j=1}^{N} s_j = \sum_{i=1}^{N} s_i \cdot x = x^2$$

$$\mathcal{Z} = \sum_{s_1=\pm1}\cdots\sum_{s_N=\pm1} \exp\left\{\frac{\beta J}{2N}\left[x^2 - N\right] + \beta H x\right\} =$$

$$= \sum_{s_1=\pm1}\cdots\sum_{s_N=\pm1} e^{-\frac{\beta J}{2}} \exp\left\{\frac{\beta J}{2N} x^2 + \beta H x\right\}$$

with $\int_{-\infty}^{+\infty} dy\, e^{-ay^2+by} = \sqrt{\frac{\pi}{a}}\, e^{b^2/(4a)}$

transform $e^{\frac{\beta J}{2N} x^2}$ with $x^2 = b^2$

$$\frac{1}{4a} = \frac{\beta J}{2N}$$

$$a = \frac{N}{2\beta J} \quad , \quad \sqrt{\frac{a}{\pi}} = \sqrt{\frac{N}{2\pi\beta J}}$$

therefore
$$e^{\frac{\beta J}{2N} x^2} = \sqrt{\frac{N}{2\pi\beta J}} \int dy \exp\Big\{ -\frac{N}{2\beta J} y^2 + y x \Big\}$$

$y$ an auxilliary field

$$Z = e^{-\frac{\beta J}{2}} \sum_{S_1 = \pm 1} \cdots \sum_{S_N = \pm 1} \sqrt{\frac{N}{2\pi\beta J}} \int dy \, e^{-\frac{N}{2\beta J} y^2 + y x + \beta H x}$$

Hubbard-Stratonovich Transformation

exponent now linear in $x$, sum over spin states can be performed

$$\sum_{S_1 = \pm 1} \cdots \sum_{S_N = \pm 1} \cdots = \prod_{j=1}^{N} \Big( \sum_{S_j = \pm 1} \cdots \Big)$$

$$Z = e^{-\frac{\beta J}{2}} \sqrt{\frac{N}{2\pi\beta J}} \int dy \, e^{-\frac{N}{2\beta J} y^2} \{ 2\cosh(y + \beta H) \}^N$$

define function

$$f(y) := \frac{1}{2\beta J} y^2 - \ln\{ 2\cosh[y + \beta H] \}$$

and observe $\sqrt{N} = e^{\frac{1}{2}\ln N}$

$$Z = \frac{e^{-\frac{\beta J}{2}}}{\sqrt{2\pi\beta J}} \lim_{N \to \infty} \int dy \, e^{-N f(y) + \frac{1}{2}\ln N}$$

observation: integral well approximated for large $N$ by value at minimum of $f(\eta)$ for $\eta = \eta^*$

$$\partial_\eta f(\eta)\Big|_{\eta=\eta^*} \overset{!}{=} 0 \quad \Rightarrow \frac{2\eta^*}{2\beta J} = \frac{2\sinh(\eta^*+\beta H)}{2\cosh(\eta^*+\beta H)}$$

$$\Rightarrow \eta^* = \beta J \tanh(\eta^*+\beta H)$$

$$Z = \frac{e^{-\frac{\beta J}{2}}}{\sqrt{2\pi\beta J}}\, e^{-f(\eta^*)}$$

saddle-point approximation