2d Ising Model: Low- & High-Temperature Expansion

$$\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j$$

on a square lattice with coordination number $z=4$

Expectation:

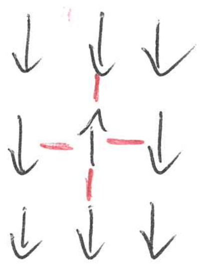
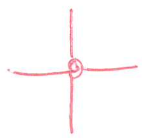
- low T: ferromagnetic, magnetization $|M|>0$
- high T: paramagnetic, $M=0$

Low Temperature Expansion

Ground State

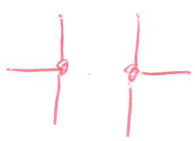
$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow \\ \downarrow & \downarrow & \downarrow \end{array} \qquad \text{Energy } E_0 = -\frac{N}{2} z J = -2NJ$$

Excitations of the ground state

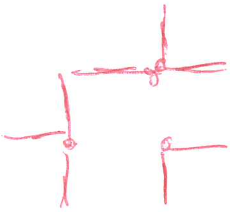
 $\equiv$ 

$E_1 = E_0 + 4 \cdot 2J = E_0 + 8J$

multiplicity N



$E_2 = E_0 + 12J$, multiplicity $2N$



$E_3^{(1)} = E_0 + 16J$

multiplicity $4N$

+ more terms @ $16J$

+ higher orders

$$Z_{tor} = \sum_{\{s_i\}} e^{\beta J \sum_{\langle ij\rangle} s_i s_j} = 2 e^{\overbrace{2N\beta J}^{E_0}}\left(1 + N e^{-\overbrace{8\beta J}^{E_1}} + 2N e^{-\overbrace{12\beta J}^{E_2}} + \frac{1}{2}(N^2+9N)e^{-\overbrace{16\beta J}^{E_3}} + \dots\right)$$

2 ground states

$$= 2e^{2N\beta J}\left(1 + N x^4 + 2N x^6 + \frac{1}{2}(N^2+9N)x^8 + \dots\right)$$

with $x := e^{-2\beta J}$

High Temperature Expansion

Rewrite $e^{\beta J s_i s_j} = \cosh \beta J + s_i s_j \sinh \beta J$

$$= \cosh \beta J (1 + s_i s_j \tanh \beta J)$$

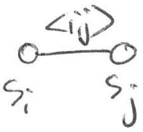
now $\tanh \beta J = \beta J + \mathcal{O}(\beta J)^3$

so expansion in βJ is equivalent to expansion in $\tanh \beta J$

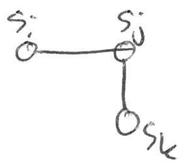
$$Z_{high} = \sum_{s_1} \cdots \sum_{s_N} \prod_{\langle ij \rangle} \cosh\beta J \,(1 + s_i s_j \tanh\beta J)$$

$$= (\cosh\beta J)^{\frac{2N}{2}} \sum_{s_1} \cdots \sum_{s_N} \prod_{\langle ij \rangle} (1 + s_i s_j \tanh\beta J)$$

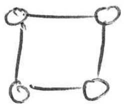
1st term $\sum_{s_1} \cdots \sum_{s_N} s_i s_j \tanh\beta J = 0$



2nd term $\sum_{s_1} \cdots \sum_{s_N} s_i s_j^2 s_k \tanh^2\beta J = 0$



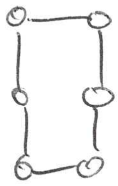
1st nonzero term $\sum_{s_1} \cdots \sum_{s_N} s_i^2 s_j^2 s_k^2 s_l^2 \tanh^4\beta J$

 $\equiv G_1(4)$

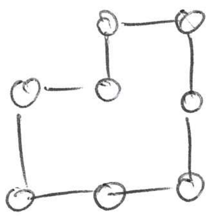
$G(L) = \{G_i(L)\}$: set of closed graphs of length L

$$Z_{high} = (\cosh\beta J)^{2N} \underbrace{\sum_{s_1} \cdots \sum_{s_N}}_{2^N} \Big[1 + \sum_{l=2}^{N} (\tanh\beta J)^{2l} \underbrace{\sum_{G_i(2l)} \underbrace{\prod_{\langle ij \rangle \in G_i(2l)} s_i s_j}_{\equiv 1}}_{|G(L)|\text{ : number of graphs of length } L} \Big]$$

$$= 2^N (\cosh\beta J)^{2N} \Big[1 + \sum_{l=2}^{N} |G(2l)| (\tanh\beta J)^{2l} \Big]$$



length $L=6$
multiplicity $2N$



length $L=8$
multiplicity $4N$

+ more @ length $L=8$
+ higher orders

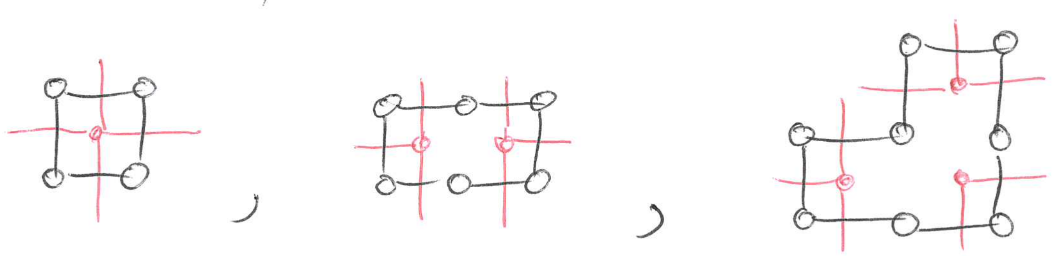
$$Z_{high} = 2^N (\cosh \beta J)^{2N} \left[1 + N (\tanh \beta J)^4 + 2N (\tanh \beta J)^6 + \tfrac{1}{2}(N^2 + 9N)(\tanh \beta J)^8 + \ldots \right]$$

$$= 2^N (\cosh \beta J)^{2N} \left[1 + N y^4 + 2N y^6 + \tfrac{1}{2}(N^2+9N) y^8 + \ldots \right]$$

with $y := \tanh \beta J$

Observation: Series in x for $Z_{low} \triangleq$ Series in y for Z_{high}

Graphically



Duality of these graphs can be proven to hold exactly

$$\frac{Z_{high}[\beta]}{2^N(\cosh\beta J)^{2N}} = \frac{Z_{low}[\beta']}{2e^{2N\beta' J}}$$

iff $\tanh\beta J = e^{-2\beta' J}$

$\Leftrightarrow \sinh 2\beta J = \frac{1}{\sinh 2\beta' J}$

Kramers-Wannier-Duality

- relates low-temperature behavior to high-temperature behavior (of the same model!)
- can also be read as duality between weak-coupling and strong-coupling

$$\sinh 2\beta J = \frac{1}{\sinh 2\beta J'}$$

- self-duality special for square lattice in 2d. Generally the dual model is something else. Still can be very useful

Z_{low} describes ferromagnetic phase
Z_{high} describes paramagnetic phase

$\Rightarrow$ Phase transition where $\beta = \beta'$

$$(\sinh 2\beta_c J)^2 = 1$$

$$\Leftrightarrow \quad k_B T_c = \frac{2J}{\ln(1+\sqrt{2})} \approx 2.27\, J$$

Exact Onsager solution confirms this prediction

Mean-Field-Approximation

Ising model on lattice with coordination z

$$\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j - H \sum_i s_i$$

Magnetization $M = \langle s_i \rangle = \frac{1}{N} \sum_i s_i$

Expand

$$s_i s_j = (s_i + M - M)(s_j + M - M) = -M^2 + M(s_i + s_j) + (s_i - M)(s_j - M)$$

therefore

$$\mathcal{H} = \frac{Nz}{2} J M^2 - JM \sum_{\langle ij \rangle} (s_i + s_j) - J \underbrace{\sum_{\langle ij \rangle} (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle)}_{\propto \Delta M^2} - H \sum_i s_i$$

Mean-Field-Approximation:

Ignore fluctuations $\Delta M^2 \stackrel{!}{=} 0$

$$\mathcal{H} \approx \mathcal{H}_{MFT} = \frac{Nz}{2} J M^2 - (H + JzM) \sum_i s_i$$

Exactly $J \sum_{\langle ij \rangle} s_i s_j = \sum_i H_i s_i$

with field $H_i = J \sum_{j \text{ nb of } i} s_j$

Replace field $H_i \rightarrow$ Mean Field JzM

Partition Function

$$Z_{MFT} = \sum_{s_1} \cdots \sum_{s_N} e^{-\beta \mathcal{H}_{MFT}} = e^{-Nz\beta J M^2/2} \sum_{s_1} \cdots \sum_{s_N} e^{\beta (H + JzM) \sum_i s_i}$$

$$= e^{-Nz\beta J M^2/2} \left[2\cosh(\beta H + \beta z J M) \right]^N$$

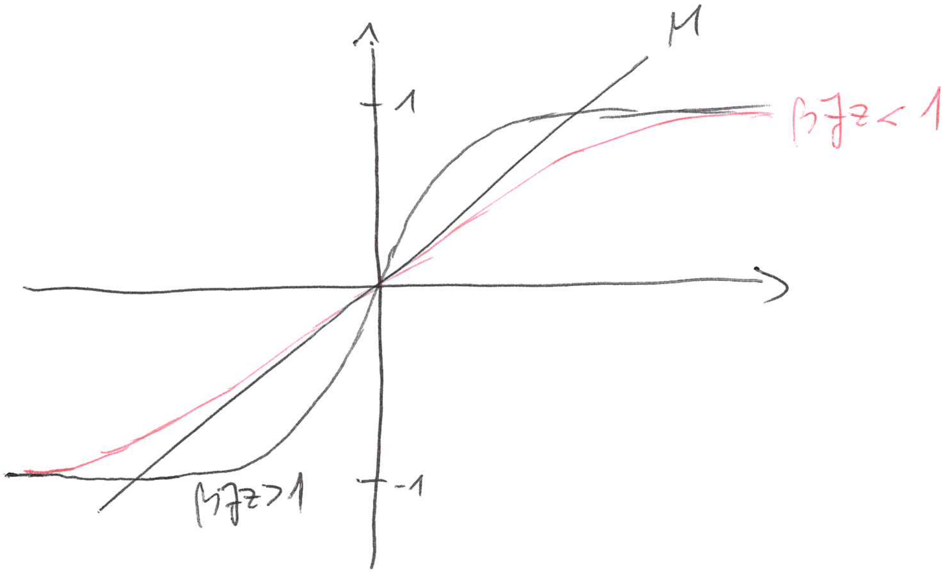
Magnetization

$$M = \frac{1}{N\beta} \frac{\partial \ln Z}{\partial H} = \tanh(\beta H + \beta J z M)$$

Self-Consistency-Equation

Zero-Field $H=0$

$$M = \tanh(\beta J z M)$$



Paramagnetic $M=0$ only solution for $\beta J z < 1$

Ferromagnetic $|M|>0$ solutions possible for $\beta J z > 1$

Susceptibility at $M=0$

$$k_B T \chi = \frac{1}{1-\beta J z} = \begin{cases} > 0 & \text{for } \beta J z < 1 \\ < 0 & \beta J z > 1 \end{cases}$$

$\Rightarrow$ State $M=0$ unstable for $\beta J z > 1$

$\Rightarrow$ Phase Transition @ $k_B T_c^{MFT} = J z$

for 2d-lattice $k_B T_c^{MFT} = 4J > 2.27J = k_B T_c^{exact}$