and thus

$$G(r) \sim -\frac{k_B T}{2\kappa} \ln r \quad \text{in } 2d$$

and

$$G(r) \sim \frac{k_B T}{2\kappa} \frac{1}{r^{d-2}} \quad \text{in } d>2$$

The Ginzburg Criterion

When is our assumption that fluctuations in H are negligible actually satisfied?
Correlations are relevant in regions of size $\sim \xi$ with volume $V = \xi^d$. The fluctuations of these correlated regions should be small compared to the magnetization of these regions

$$\sum_{i \in V} \langle (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle) \rangle \ll \left(\sum_{i \in V} s_i \right)^2$$

In our continuum language the condition reads

$$\int_V d\underline{r}\, G(r) \ll \int_V d\underline{r} \langle M(\underline{r})\rangle^2 \sim \xi^d \langle M\rangle^2$$

Because $G(r \gg \xi) \to 0$ exponentially, we can just as well integrate over all space.

In the ordered phase with $\langle M\rangle \neq 0$ we thus have

$$\int d\underline{r}\, G(r) = k_B T_c \chi = \frac{k_B T_c}{-4At} = \frac{k_B T_c \xi^2}{2\kappa}$$

For the other side of the relation we find

$$\xi^d \langle M\rangle^2 = -\frac{At}{2B}\xi^d = \frac{\kappa}{4B}\xi^{d-2}$$

and thus the condition

$$\boxed{\xi^{4-d} \ll \frac{\kappa^2}{k_B T_c B}}$$

For dimensions $d>4$ the Ginzburg criterion will always be fulfilled close enough to T_c and we will observe mean-field behavior.
For $d<4$ fluctuations will eventually become very large and mean-field theory fails.
We thus found the upper critical dimension $D_c = 4$ for the Ising model.
With $\chi \sim |t|^{-\gamma}$ we can write the Ginzburg criterion also as

$$|t|^{4-d} \gg \frac{k_B T_c B}{A^{4-d} \kappa^2} =: t_G^{4-d}$$

If $t_G \ll 1$ we will observe mean-field behavior for $t_G \ll t \ll 1$ even for $d < D_c$
This is not the case for magnetic systems where $t_G \sim 1$

But for conventional superconductors

$$t_G \sim (T_c/E_F)^4 \sim 10^{-14}$$

because the Fermi-energy E_F is much larger than the critical temperature T_c

More generally we have

$$\int d\underline{r}\, G(r) \sim k_B T_c \chi \sim |t|^{-\gamma}$$

$$\int d\underline{r} \langle M(\underline{r})\rangle^2 \sim \xi^d |t|^{2\beta} \sim t^{2\beta - \nu d}$$

such that the upper critical dimension is given by

$$D_c = \frac{2\beta + \gamma}{\nu}$$

Interlude: Linear Response in Fluids

With the correlation function

$$G(\underline{r}_1-\underline{r}_2) = \frac{1}{\rho^2}\left[\langle\rho(\underline{r}_1)\rho(\underline{r}_2)\rangle - \langle\rho(\underline{r}_1)\rangle\langle\rho(\underline{r}_2)\rangle\right]$$

we find

$$V\int d\underline{r}\, G(\underline{r}) = \int d\underline{r}_1 \int d\underline{r}_2\, G(\underline{r}_1-\underline{r}_2) = \frac{1}{\rho^2}\left[\langle N^2\rangle - \langle N\rangle^2\right]$$

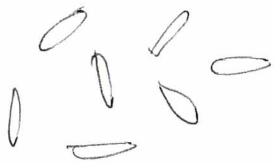
From the sum rule $k_B T \kappa_T = \int d\underline{r}\, G(\underline{r})$ we obtain

$$\Delta N^2 \equiv \langle N^2\rangle - \langle N\rangle^2 = k_B T \rho^2 V \kappa_T$$

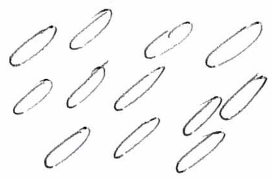
On approach to the critical point the correlation length ξ grows. Via the sum rule this implies a growth in the compressibility (we also know that the susceptibility diverges). Finally, the equation above tells us that the fluctuations in particle number will diverge @ T_c

Liquid Crystals: The Isotropic-Nematic Transition

Consider a fluid of apolar elongated particles



Nematic Phase: Particles at random positions but with aligned orientations



Nematic Order Parameter: Should be zero in the fluid phase and nonzero in the nematic phase.

Introduce unit vector $\underline{u}_i$ parallel to long axis of the particle.

$\langle \underline{u}_i \rangle$ is zero in both phases because $\underline{u}_i$ is equivalento to $-\underline{u}_i$ (apolar)

Simplest order parameter is a tensor (matrix)

$$S_{\alpha\beta} = \frac{1}{N} \sum_i \left(u_i^\alpha u_i^\beta - \frac{1}{3} \delta_{\alpha\beta} \right)$$

Landau free energy

$$\mathcal{F}[\underline{\underline{S}}, T) = F_0(T) + \frac{1}{2} A (T - T^*) \mathrm{Tr}(\underline{\underline{S}}^2) - \frac{1}{3} B \, \mathrm{Tr}(\underline{\underline{S}}^3)$$

$$+ \frac{1}{4} C' \, \mathrm{Tr}^2(\underline{\underline{S}}^2)$$

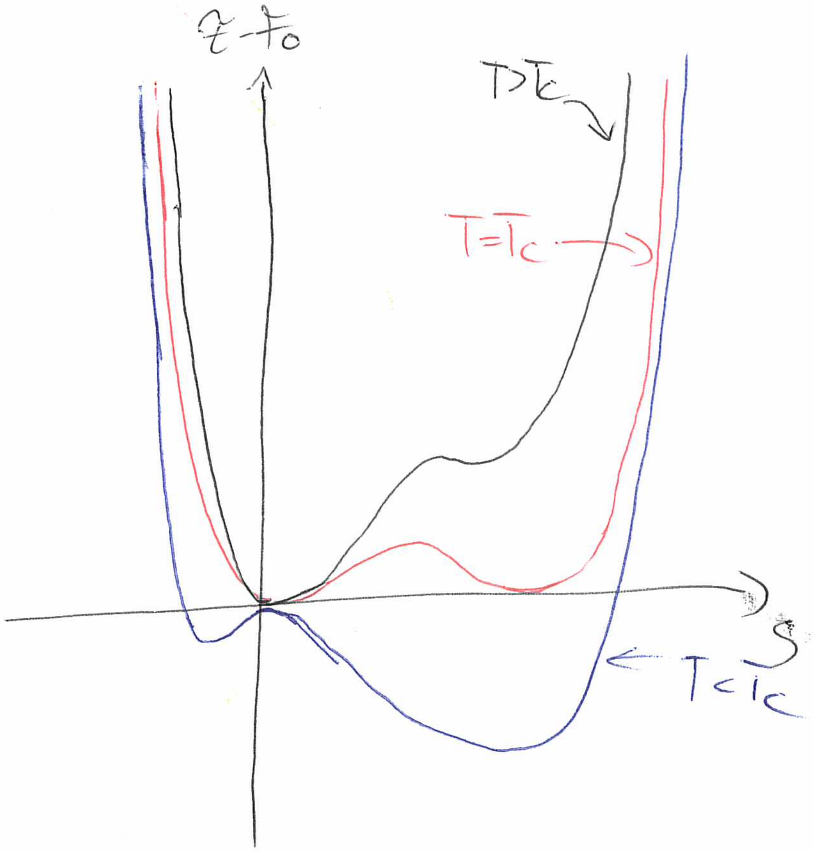
For the isotropic-nematic transition we can write

$$S_{\alpha\beta} = S \left(n_\alpha n_\beta - \frac{1}{3} \delta_{\alpha\beta} \right)$$

with the (global) director $\underline{n}$. Then

$$\mathcal{F}(S, T) = F_0(T) + \frac{1}{3} A (T - T^*) S^2 - \frac{2}{27} B S^3 + \frac{1}{9} C S^4$$

We are looking for the global minimum $\bar{S}$ of $\mathcal{F}$.



The global minimum $\bar{\xi}$ jumps discontinuously from $\bar{\xi}=0$ above T_c to $\bar{\xi}_c = \frac{B}{3C} > 0$ at the critical temperature $T_c \neq T_*$.

The third-order term in the Landau free energy generated a discontinuous or first-order phase transition.

For this to be valid, we need $\bar{\xi}_c \ll 1$ and thus Landau Theory only applies to weakly first order transitions

The XY-Model : Part I

Consider two-dimensional vector spins $\underline{s}_i = \begin{pmatrix} \cos\varphi_i \\ \sin\varphi_i \end{pmatrix}$ on a d-dimensional lattice with lattice spacing a

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \underline{s}_i \cdot \underline{s}_j = -J \sum_{\langle ij \rangle} \cos(\varphi_i - \varphi_j)$$

With the magnetization

$$\underline{M} = \frac{1}{N} \sum_i \underline{s}_i = \langle \underline{s}_i \rangle = M \begin{pmatrix} \cos\Theta \\ \sin\Theta \end{pmatrix}$$

we can immediately write down the Landau free energy

$$\tilde{F}(M,T) = F_0(T) + At\,\underline{M}^2 + B\,\underline{M}^4$$

which predicts a ferromagnetic phase $\underline{M} \neq \underline{0}$ below the critical temperature T_c and places the model in the Ising universality class.

In the ordered phase, neighbouring spins will be almost aligned and we can approximate

$$\mathcal{H} \approx -J \sum_{\langle ij \rangle} \left[1 - \frac{1}{2}(\varphi_i - \varphi_j)^2\right] = E_0 + \frac{1}{2} J \sum_{\langle ij \rangle} (\varphi_i - \varphi_j)^2$$

$$= E_0 + \frac{1}{2} J a^2 \sum_{\langle ij \rangle} \left(\frac{\varphi_i - \varphi_j}{a}\right)^2 \to E_0 + \frac{1}{2} J a^{2-d} \int d\underline{r} \, (\nabla \theta(\underline{r}))^2$$

upon coarse-graining

In Fourier space

$$\theta(\underline{r}) = \frac{1}{(2\pi)^{d/2}} \int d\underline{k} \, e^{i\underline{k}\cdot\underline{r}} \, \theta(\underline{k})$$

and with

$$\frac{1}{(2\pi)^d} \int d\underline{r} \, e^{i(\underline{k}_1 + \underline{k}_2)\cdot\underline{r}} = \delta(\underline{k}_1 - \underline{k}_2)$$

we have

$$\mathcal{H} = \frac{J}{2} \frac{a^{2-d}}{(2\pi)^d} \int d\underline{r} \int d\underline{k}_1 \int d\underline{k}_2 \, e^{i\underline{k}_1\cdot\underline{r}} e^{i\underline{k}_2\cdot\underline{r}} \, i\underline{k}_1 \cdot i\underline{k}_2 \, \theta(\underline{k}_1)\theta(\underline{k}_2)$$

$$= \frac{\tilde{J}}{2} a^{2-d} \int d\underline{k} \, k^2 \, \theta(\underline{k})\theta(-\underline{k})$$

$$\to \frac{1}{2} J \sum_{\underline{k}} k^2 |\theta(\underline{k})|^2$$

Then the partition function

$$Z = e^{-\beta E_0} \int \prod_{\substack{k \\ k<a}} d\theta(\underline{k})\, e^{-\frac{1}{2}\beta J \sum_{\underline{k}} k^2 |\theta(\underline{k})|^2}$$

is just a product of Gaussian integrals. To compute the mean magnetization

$$\langle M_x \rangle = M \langle \cos\theta(0) \rangle = \frac{M}{Z} \mathrm{Re} \int \mathcal{D}[\theta(\underline{x})]\, e^{-\beta \mathcal{H} + i\theta(0)}$$

we can go to the fourier domain and complete the square in the exponent. From the correction factor we find

$$\langle M_x \rangle \sim \exp\left[-\frac{c}{\beta J} \int d\underline{k} \frac{1}{k^2}\right] = \exp\left[-\frac{c'}{\beta J} \int_{\pi/L}^{\pi/a} dk\, k^{d-3}\right]$$

We are going to need the integral

$$I(a, L) := \int_{\pi/L}^{\pi/a} dk\, k^{d-3}$$

depending on the lattice spacing a and the system size L a lot.

We have

$$I(a, L\to\infty) \sim \begin{cases} L & \\ \ln\frac{L}{a} & \text{for } d=2 \\ a^{2-d} = \text{const.} & d>2 \end{cases} \quad \begin{matrix} d=1 \\ \\ \\ \end{matrix}$$

This implies for large systems $L\to\infty$

$$\langle M_x\rangle \sim \begin{cases} \text{const.} & d>2 \\ a/L \to 0 \quad \text{for} & d=2 \\ e^{-L}\to 0 & d=1 \end{cases}$$

So our assumption of an ordered phase with $\langle \underline{M}\rangle \neq \underline{0}$ is actually only consistent for $d>2$, i.e. the lower critical dimension is $d_c=2$

We can corroborate this result by looking at how well different parts of the system are correlated

$$\langle \underline{M}(\underline{r})\cdot\underline{M}(\underline{0})\rangle \sim \langle \cos(\theta(\underline{r})-\theta(\underline{0}))\rangle = \mathrm{Re}\langle e^{i(\theta(\underline{r})-\theta(0))}\rangle$$

$$= \mathrm{Re}\exp\left[-\tfrac{1}{2}\langle(\theta(\underline{r})-\theta(\underline{0}))^2\rangle\right]$$

where the last identity holds because we have a quadratic Hamiltonian. Now

$$\langle(\theta(\underline{r})-\theta(\underline{0}))^2\rangle = \frac{1}{(2\pi)^d}\int d\underline{k}_1\int d\underline{k}_2\,(e^{i\underline{k}_1\cdot\underline{r}}-1)(e^{i\underline{k}_2\cdot\underline{r}}-1)$$

$$\times\langle\theta(\underline{k}_1)\,\theta(\underline{k}_2)\rangle$$

Moreover

$$\langle\theta(\underline{k}_1)\theta(\underline{k}_2)\rangle = \frac{1}{Z}\int\mathcal{D}\theta\,\theta(\underline{k}_1)\theta(\underline{k}_2)\,e^{-\beta\mathcal{H}}$$

$$= \frac{\delta(\underline{k}_1+\underline{k}_2)}{\beta J k_1^2}$$

because it is again just a fancy Gaussian integral

$$\langle(\theta(\underline{r})-\theta(\underline{0}))^2\rangle = \frac{1}{(2\pi)^d\beta J}\int\frac{d\underline{k}}{k^2}(e^{i\underline{k}\cdot\underline{r}}-1)(e^{-i\underline{k}\cdot\underline{r}}-1)$$

$$= \frac{2}{(2\pi)^d\beta J}\int d\underline{k}\,\frac{1-\cos\underline{k}\cdot\underline{r}}{k^2}$$

We have

$$\int d\underline{k} \, \frac{1-\cos \underline{k}\cdot\underline{r}}{k^2} \sim \int_{\pi/L}^{\pi/a} dk \, \frac{1-\cos kr}{k^{d-3}}$$

$$= \underbrace{\int_{\pi/L}^{\pi/r} dk \, \frac{1-\cos kr}{k^{d-3}}}_{=: I_1} + \underbrace{\int_{\pi/r}^{\pi/a} dk \, \frac{1-\cos kr}{k^{d-3}}}_{=: I_2}$$

For $d \geq 2$ we find

$$I_1 \leq 2 \int_{\pi/L}^{\pi/r} dk \, \frac{1}{k^{d-3}} = I(r,L) \to 0 \quad \text{for } r \to \infty$$

The cos in I_2 will be rapidly oscillating and therefor average out

$$I_2 \sim I(a,r) \sim \begin{cases} a^{2-d} = \text{const.} & \\ \ln \frac{r}{a} & \end{cases} \text{for} \begin{matrix} d=2 \\ d>2 \end{matrix}$$

i.e.

$$\langle (\theta(\underline{r}) - \theta(\underline{0}))^2 \rangle \xrightarrow{r\to\infty} \begin{cases} \text{const.} & \\ \sim \frac{1}{\beta J} \ln \frac{r}{a} & \end{cases} \text{for} \begin{matrix} d>2 \\ d=2 \end{matrix}$$

$$\langle \underline{M}(\underline{r}) \cdot \underline{M}(\underline{0}) \rangle \xrightarrow{r\to\infty} \begin{cases} \text{const} & \\ \sim \left(\frac{r}{a}\right)^{-\eta} \to 0 & \end{cases} \text{for} \begin{matrix} d>2 \\ d=2 \end{matrix}$$