where $\eta = \frac{1}{2\pi\beta J}$

The O(n)-model and Spontaneous Symmetry Breaking

Consider d-dimensional spins $\underline{s}_i$ on a d-dimensional lattice

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \underline{s}_i \cdot \underline{s}_j$$

The case d=3 is called the Heisenberg-Model. Note that the Hamiltonian is invariant under rotations in the d-dimensional spin-space, i.e. it posesses a continuous symmetry. The Landau free energy functional reads

$$\mathcal{F} = \int d\underline{r} \left[\frac{1}{2} a \underline{M}^2 + \frac{1}{4} B \underline{M}^4 + \frac{1}{2} (\nabla \cdot \underline{M})^2 \right]$$

where $\underline{M}(\underline{r})$ is the coarse-grained magnetization and we set $\kappa = 1$
Below the critical temperature T_c we expect an ordered phase with $\underline{M} = \bar{M}\underline{n}$, $\bar{M} \neq 0$ and an arbitrary direction $\underline{n}$ which is said to have spontaneously broken the rotational symmetry.
Let's consider perturbations of the magnetization

$$\underline{M} \to \underline{M}(\underline{r}) = \bar{M}(\underline{n} + \underline{\phi}(\underline{r}))$$

and decompose $\underline{\phi}$ into components parallel and perpendicular to $\underline{n}$

$$\underline{\phi} = \phi_{\parallel}\underline{n} + \underline{\phi}_{\perp}$$

We can then expand $\mathcal{F}$ to second order in $\underline{\phi}$ and find

$$\mathcal{F}[\underline{M}] = \mathcal{F}[\bar{M}\underline{n}] + \mathcal{F}[\underline{\phi}] + \mathcal{O}(\phi^3)$$

where

$$\mathcal{F}[\underline{\phi}] = \frac{\hbar^2}{2}\int d\underline{r}\,\left[(\nabla\underline{\phi}_\perp)^2 + (\nabla\phi_\parallel)^2 + 2|a|\phi_\parallel^2\right]$$

$$= \mathcal{F}[\underline{\phi}_\perp] + \mathcal{F}[\phi_\parallel]$$

and in Fourier space

$$\mathcal{F}[\underline{\phi}] = \hbar^2\int d\underline{k}\,\left[k^2|\underline{\phi}_\perp|^2 + k^2|\phi_\parallel|^2 + 2|a||\phi_\parallel|^2\right]$$

Regarding $\mathcal{F}[\underline{\phi}]$ as the Hamiltonian of the field $\underline{\phi}(\underline{r})$ we see that the excitations decompose into one longitudinal mode $\phi_\parallel$ and d-1 Goldstone modes that cost arbitrarily low energy for $k \to 0$

Remembering the equation for the susceptibilities

$$\frac{\delta^2\mathcal{F}}{\delta\phi_\alpha\,\delta\phi_\beta}\,\chi_{\alpha\beta} = \delta(\underline{r}_1 - \underline{r}_2)$$

or in Fourier space

$$\frac{\delta^2 \mathcal{H}}{\delta\phi_\alpha(\underline{k})\delta\phi_\beta(-\underline{k})}\tilde{\chi}_{\alpha\beta}(\underline{k}) = 1$$

we find

$$\tilde{\chi}_{\parallel}^{-1} \sim 2|a| + k^2 , \qquad \tilde{\chi}_{\perp}^{-1} \sim k^2$$

Moreover $\chi(r) \sim e^{-r/\xi}$ corresponds to $\tilde{\chi}(k) \sim \frac{1}{\xi^{-2}+k^2}$, i.e. the Goldstone modes have an infinite correlation length.

$$\mathcal{H}[\phi_\perp] = \frac{1}{2}\bar{M}^2 \int d\underline{r}\,(\nabla\phi_\perp)^2$$

is identical to the Hamiltonian for the perturbations of the ordered phase in the XY-model. We conclude that the Goldstone modes will not be long-range correlated in $d \leq 2$ and will destroy the ordered phase. This can be proven in greater generality, leading to the

Mermin-Wagner Theorem:

The lower critical dimension for continuous symmetry breaking is $d_c = 2$

XY-Model, Part II: The Kosterlitz-Thouless Transition

Helium-4 has a phase transition to a superfluid phase at the critical lambda-temperature $T_\lambda = 2.17K$.

In the low-temperature phase, Helium-4 forms a Bose-Einstein-condensate where the phase $\Theta(\underline{r})$ of its quantum-mechanical wave function $\psi(\underline{r}) = e^{i\Theta(\underline{r})}$ is aligned over macroscopic distances.

If we regard the real- and imaginary part of the wave function as the components of a two-dimensional vector $S_x + iS_y$ we can map the problem to the XY-model. We conclude that the superfluid-transition is a spontaneous symmetry breaking of the invariance under phase-shifts and that it should occur in 3d Helium-4 bulk fluids but not in 2d Helium-4 films.
Nevertheless, experiments (see Fig. at the end of todays lecture) show indications of a phase transition in monolayers of Helium-4.

For the XY-model in 2d we have seen that the fluctuations in the order parameter prevent a divergent correlation length and instead the order parameter becomes uncorrelated over large distances as

$$\langle \underline{M}(\underline{r}) \cdot \underline{M}(\underline{0}) \rangle \sim r^{-\eta} \qquad \eta = \frac{1}{2\pi\beta J}$$

Usually, we find this critical power-law scaling of the correlation function only at the critical point. Here, it is valid for the whole low-temperature phase.
For low temperatures, the exponent η becomes very small and in a finite system we may hardly notice the decay of the correlation function

Therefor we call this a phase of quasi-long-range order.
What can destroy the quasi-long-range order to induce a phase transition?
What we missed by using the approximate Hamiltonian

$$\mathcal{H} = E_0 + \frac{1}{2} J \int d\underline{r} \, (\nabla \theta(\underline{r}))^2$$

are (topological) defects in the arrangement of spins where the gradients are no longer small but become singular. Vortices and Anti-Vortices are such topological defets (see Fig at the end)
The ground state of the vortex-free system $\frac{\delta \mathcal{H}}{\delta \theta} \stackrel{!}{=} 0$ is the solution of the Laplace equation

$$\nabla^2 \theta(\underline{r}) = 0$$

of 2d Electrodynamics.
Integrating along a closed curve around a vortex

$$\oint \nabla\Theta(\underline{r})\cdot d\underline{l} = \pm 2\pi n, \qquad n\in\mathbb{N}$$

is the equivalent of Gauss law with the "electric field" $\nabla\Theta$ and the "charge" $2\pi n$. The latter is quantized to fulfill the continuity of the angle Θ.
Integrating around a circle of radius r around the vortex we find

$$2\pi n = \oint \nabla\Theta(\underline{r})\cdot d\underline{l} = 2\pi r|\nabla\Theta(r)|$$

such that

$$|\nabla\Theta(r)| = \frac{n}{r}$$

and we find for the energy of a vortex

$$E_{vor} = E_0 + \frac{1}{2}J\int d\underline{r}\,|\nabla\Theta(\underline{r})|^2$$

$$= \pi n^2 J\int_a^L dr\,\frac{1}{r} = \pi n^2 J \ln\frac{L}{a}$$

which diverges for large systems $L \to \infty$. An isolated vortex will cost too much energy to be created.

Gauss law tells us that a pair of a vortex and an anti-vortex at a distance R will be invisible at distances $\gg R$ because the charges cancel. A pair will have an energy

$$E_{pair} = 2E_c + E \ln(R/a)$$

where E_c is the energy of the vortex core that we do not properly resolve and $E \sim J$.

We could place the vortex on any of the L^2/a^2 lattice sites such that vortices will increase the entropy by

$$\Delta S \sim k_B \ln(L^2/a^2)$$

As a result we find the change of free energy due to vortex-anti vortex pairs

$$\Delta F = E_{pair} - 2T\Delta S$$

$$= 2E_c + E \ln \frac{R}{L} + (E - 2k_BT) \ln \frac{L}{a}$$

For low temperatures $k_BT < E/2$ we will only lower the free energy if $R \sim a$, i.e. the pairs are very small and will not destroy the ordered phase.

For $k_BT > E/2$ vortex-anti vortex pairs lower the free energy even for $R \sim L$, i.e., the pair becomes essentially unbound above a critical temperature

$$T_c \sim E \sim J$$

Above T_c the system contains a certain density of free vortices with a characteristic mean distance l which limits the correlation length $\xi \sim l$.
As a result, T_c marks the critical temperature of the

Kosterlitz-Thouless-Transition

where the correlation function changes from

$$G(r) \sim r^{-\eta}$$

below T_c, to

$$G(r) \sim e^{-r/\xi} \quad \text{with } \xi < \infty$$

above T_c or from quasi-long-range order to disorder.