Renormalization

ultimate goal: - understand phase transition from critical point

- use existence of critical point for calculations

motivation by observations in experiment and numerical simulation

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Static Phenomena Near Critical Points: Theory and Experiment

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Kadanoff et al RMP (1967)
collection of experimental results

Wilson, Sci Am 241, 140 (1979)
2D-Ising simulation on square lattice with block-spin renormalization

$$\begin{matrix}\uparrow & \downarrow & \uparrow \\ \downarrow & \uparrow & \uparrow \\ \downarrow & \downarrow & \uparrow\end{matrix} \longrightarrow \uparrow$$ and so on

$T > T_c$: successive steps make pattern more randomized

$T < T_c$: already dominant spin orientation becomes amplified

$T = T_c$: pattern similar even after many successive steps of renormalization

correlation length ξ divergent (∞) at critical point

main observation

now: put that observation into a proper theory to calculate numbers

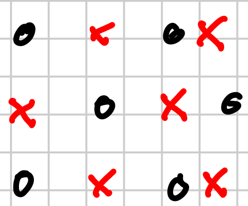
identification of fixed points of a suitable renormalization procedure with critical point(s)

reduced Hamiltonian $\bar{\mathcal{H}} = \mathcal{H}/k_BT$

new reduced Hamiltonian $\bar{\mathcal{H}}' = \mathcal{R}\,\bar{\mathcal{H}}$

$\mathcal{R}$: RNG operator

real-space scale factor b, say $b = \sqrt{2}$



new lattice with original topology but different scale

unchanged by RNG procedure

↳ partition function $Z_{N'}(\bar{\mathcal{H}}') = Z_N(\bar{\mathcal{H}})$

↳ total free energy $\mathcal{F}_{N'}(\bar{\mathcal{H}}') = \mathcal{F}_N(\bar{\mathcal{H}})$

$\frac{N}{N'} = b^d$, d dimensions, N degrees of freedom

reduced free energy per spin $\tilde{f} = f/k_BT$

$$\tilde{f}(\bar{\mathcal{H}}') = b^d$$

lengths $\vec{r} \to \vec{r}\,' = b^{-1}\vec{r}$

momenta $\vec{q} \to \vec{q}\,' = b\vec{q}$

spin vector $\vec{s}_{\vec{r}} \to \vec{s}\,'_{\vec{r}\,'} = c^{-1}\vec{s}_{\vec{r}}$

↳ pair correlation function transforms as

$$c^2\Gamma(b^{-1}\vec{r}, \bar{\mathcal{H}}') = \Gamma(\vec{r}, \bar{\mathcal{H}})$$

fixed point of RNG procedure

$$\bar{\mathcal{H}}' = \bar{\mathcal{H}} = \bar{\mathcal{H}}^*$$

correlation length in units of lattice spacing

$$\xi' = \xi = \xi^* = b^{-1}\xi \Leftrightarrow \xi = \infty$$

flow in parameter space:

$$\bar{\mathcal{H}} = \sum_\alpha \vec{\mu}\cdot\vec{f}\,, \quad \vec{f} \text{ generalized operators}, \quad \vec{\mu} \text{ conjugate fields}$$

$\vec{\mu}$: vector marking system in an infinite-dimensional space

RNG procedure makes $\vec{\mu}$ move through parameter space

$$\vec{\mu}' = R\,\vec{\mu}\,, \quad \vec{\mu}' = \vec{\mu} = \vec{\mu}^*$$

now: expansion possible around fixedpoint $\vec{\mu}^*$

$$\vec{\mu} = \vec{\mu}^* + \delta\vec{\mu}$$

$$\vec{\mu}' = \vec{\mu}^* + \delta\vec{\mu}'$$

$$\delta\vec{\mu}' = A(\vec{\mu}^*)\,\delta\vec{\mu}$$

↳ constant matrix with Eigenvalues λ_i!

Eigenvalues functions of scale factor b,

$$\lambda_i(b_1)\,\lambda_i(b_2) = \lambda_i(b_1 b_2)$$

↳ constraints on Eigenvalues

$$\lambda_i(b) = b^{y_i}$$

y_i independent of scale factor b

$$\vec{\mu} = \vec{\mu}^* + \sum_i g_i \vec{v}_i$$

g_i, linear scaling fields

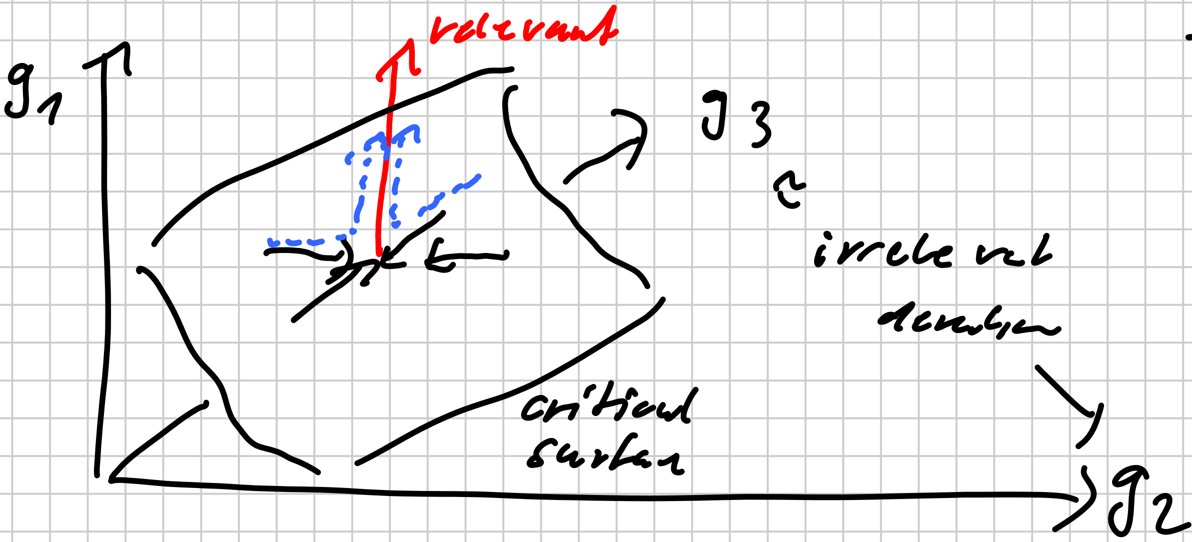
$$\vec{\mu}' = \vec{\mu}^* + \sum_i b^{y_i} g_i \vec{v}_i$$

↳ $g_i' = b^{y_i} g_i$

$y_i < 0$: corresponding field decreases under RNG → irrelevant variables

$y_i > 0$: field increases under RNG (example: temperature) → relevant variables

general trend: only a small number of relevant variables remains for description of critical surface



any point $\vec{\mu}$ with any component along a relevant scaling field is driven away from fixed point; points with relevant scaling fields being zero exactly flow into the fixed point and define the critical surface.

Scaling and Critical Exponents

$$\bar{f}(\vec{\mu}) = b^{-d}\,\bar{f}(\vec{\mu}')$$

$$\bar{f}(g_1, g_2, g_3, \dots) \sim b^{-d}\,\bar{f}(b^{y_1} g_1, b^{y_2} g_2, \dots)$$

for magnetic system $g_1 = t = \frac{T-T_c}{T_c}$, $g_2 = h$

$$\boxed{\bar{f}(t, h, g_3, \dots) \sim b^{-d}\,\bar{f}(b^{y_t} t, b^{y_h} h, \dots)} \quad (*)$$

Scaling form of free energy

Specific heat close to critical point

$$\frac{\partial^2 \bar{f}}{\partial t^2} = \bar{f}_{tt}(h=0) \sim |t|^{-\alpha}$$

2nd derivative of Eq. (*)

$$\bar{f}_{tt}(t,0) \sim b^{-d+2y_t}\,\bar{f}_{tt}(b^{y_t}t,0)$$

fix arbitrary scale factor b by setting

$$b^{y_t}|t| = 1$$

$$\bar{f}_{tt}(t,0) \sim |t|^{\frac{d-2y_t}{y_t}}\,\bar{f}_{tt}(\pm 1,0)$$

hence

$$\alpha = 2 - d/y_t$$