Critical Exponents of the Ising model

		$d > 4$	$d = 3$
$M \sim (-t)^\beta$	β	$1/2$	$0.326\ldots$
$\chi \sim \lvert t\rvert^{-\gamma}$	γ	1	$1.237\ldots$
$M \sim H^{1/\delta}$	δ	3	$4.789\ldots$
$\xi \sim \lvert t\rvert^{-\nu}$	ν	$1/2$	$0.629\ldots$

For $d > 4$ from the Landau Theory and for $d=3$ from simulations.

Observation: The critical laws still hold in $d=3$ where the mean-field treatment fails but the exponents attain weired values that are not obviously any rational number

How can these exponents appear?
Let us look at the dimensions of the terms in the Landau Free Energy

$$\beta \mathcal{F} = \int d^d r \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} a \phi^2 + \frac{1}{4} B \phi^4 \right]$$

The "Boltzmann-factor" $\beta\mathcal{F}$ is dimensionless, so we write $[\beta\mathcal{F}]=1$ and denote the dimension of length by L such that $[r]=L$. Then the integral must also be dimensionless, i.e. term by term

$$\left[\int d^d r (\nabla\phi)^2 \right] = L^d \cdot L^{-2} [\phi]^2 \stackrel{!}{=} 1$$

$$\Rightarrow [\phi] = L^{1-d/2}$$

$$\left[\int d^d r \, a \phi^2 \right] = L^d \cdot L^{2-d} \cdot [a] \stackrel{!}{=} 1$$

$$\Rightarrow [a] = L^{-2}$$

$$\left[\int d^d r\, B\phi^4\right] = L^d L^{4-2d} [B] \doteq 1$$

$$\Rightarrow [B] = L^{d-4}$$

So $1/\sqrt{a} = 1/\sqrt{At} =: L$ is a natural candidate for a lengthscale.
But the correlation length
$\xi \sim L \sim t^{-1/2}$ so we have $\nu = 1/2$
necessarily from these dimensional arguments in contradiction to the measurements for $d=3$.
The only other lengthscale we have is the lattice spacing ℓ. The correlation length could (and does) also depend on the ratio ℓ/L, i.e.

$$\xi \sim L g(\ell/L)$$

with some function $g(x)$

Close to T_c, the length scale L diverges so it seems to be safe to replace $g(x)$ by $g(0)$ for $x \to 0$

Anomalous Dimensions

We cannot set $g(x\to 0) = g(0)$ if g diverges like $g(x\to 0) \sim x^{-\theta}$.
Then we have

$$\xi \sim L\, g(\ell/L \to 0) \sim L \frac{\ell^{-\theta}}{L^{-\theta}} = L^{1+\theta}\ell^{-\theta}$$

and with $L \sim t^{-1/2}$

$$\xi \sim t^{-\frac{1+\theta}{2}} \ell^{-\theta} \sim t^{-\nu}$$

with $\nu = \frac{1+\theta}{2} \neq \frac{1}{2}$

The exponent θ is called an <u>anomalous dimension</u> and it allows for a nontrivial critical exponent $\nu \neq 1/2$