Calculation of Critical Exponents

scaling form of the free energy

$$\boxed{\bar{f}(t,h) \sim b^{-d} \bar{f}(b^{y_t} t, b^{y_h} h)}$$

exponent α for specific heat

$$C \sim \partial_t^2 \bar{f} \sim |t|^{-\alpha}$$

$$\partial_t^2 \bar{f} \sim b^{-d} b^{y_t} b^{y_t} \partial_t^2 \bar{f}(b^{y_t} t, b^{y_h} h)$$
$$= b^{-d+2y_t} \partial_t^2 \bar{f}(b^{y_t} t, b^{y_h} h) = *$$

b is arbitrary $\to$ set to something convenient

$$b^{y_t} |t| = 1 \,, \quad b = |t|^{-1/y_t}$$

$$* = \left(|t|^{-1/y_t}\right)^{-d+2y_t} \partial_t^2 \bar{f}(\pm 1, 0)$$
$$= |t|^{d/y_t - 2} \stackrel{!}{=} |t|^{-\alpha} \Rightarrow \boxed{\alpha = 2 - \frac{d}{y_t}}$$

exponent β for magnetization

$$M \sim \partial_h \bar{f} \sim -t^{\beta}$$

$$\partial_h \bar{f} \sim b^{-d} b^{y_h} \partial_h \bar{f}(b^{y_t} t, b^{y_h} h)$$

and fixing b as above

$$\partial_h \bar{f} \sim |t|^{-\frac{d-y_h}{y_t}} \stackrel{!}{=} -t^{\beta} \Rightarrow \boxed{\beta = \frac{d-y_h}{y_t}}$$

exponent γ for susceptibility

$$\chi_T \sim -\partial_h^2 \bar{f} \sim b^{-d} b^{2y_h} \partial_h^2 \bar{f}$$

$$\partial_h \bar{f} \sim |t|^{-\frac{2y_h - d}{y_t}} \rightarrow \boxed{\gamma = \frac{2y_h - d}{y_t}}$$

exponent δ for critical isotherm ($t=0$)

$$h \sim |M|^{\delta} \operatorname{sgn}(M)$$

$$\partial_h \bar{f} = b^{-d} b^{y_h} \partial_h \bar{f}(0, b^{y_h} h)$$

fix $b^{y_h} h = 1$ and $\partial_h \bar{f} \sim h^{\frac{d-y_h}{y_h}} = h^{\frac{1}{\delta}}$

$$\rightarrow \boxed{\delta = \frac{y_h}{d-y_h}}$$

all together 4 exponents expressed by two variables y_t and y_h

$\hookrightarrow$ 2 relations among exponents

$$\alpha + 2\beta + \gamma = 2$$

$$\gamma = \beta(\delta - 1)$$

for ν and η, consider

scaling form of pair correlation function

$$\boxed{\Gamma(\vec{r}, t, h) \sim c^2(b)\, \Gamma(b^{-1}\vec{r}, b^{y_t} t, b^{y_h} h)}$$

at zero field ($h=0$) and fixing $b^{y_t}|t| = 1$

$$\Gamma(\vec{r}, t) \sim c^2(|t|^{-1/y_t})\, \Gamma(|t|^{-1/y_t}\vec{r}, \pm 1)$$

exponent ν for correlation length

correlation length $\xi \sim |t|^{-\nu}$

scales like $b \sim |t|^{-1/y_t} \Rightarrow \boxed{\nu = 1/y_t}$

exponent η for pair correlation function at critical point

$$\Gamma(\vec{r}, 0) \sim r^{-(d-2+\eta)}$$

hence $c(b) = b^{-(d-2+\eta)/2}$ as prefactor

for the susceptibility χ_T we have

$$\chi_T \sim \int dr\, \Gamma(r)\, r^{d-1}$$

substitute $r = x\xi$

$$\chi_T \sim \xi^{2-\eta}$$

$$t^{-\gamma} \sim t^{-(2-\eta)\nu} \Rightarrow \gamma = (2-\eta)\nu$$

$$\Rightarrow \boxed{\eta = 2 + d - 2y_h}$$

RNG procedure, scaling form of free energy and correlation functions motivates the use of scaled variables for plotting experimental data magnetic field and magnetization for ferromagnet $CrBr_3$

Ho/Lister, Phys. Rev. Lett. 22, 603 (1969)

RNG treatment of 1D Ising model

$$\bar{\mathcal{H}} = \beta \mathcal{H} = -K \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i - E \sum_i 1$$

1D lattice

original	1	2	3	4	5	6	7	8
decimation	1	$\cancel{2}$	3	$\cancel{4}$	5	$\cancel{6}$	7	$\cancel{8}$
new	1	2	3	4	5	6	7	8

details of calculations in homework sheets, general route of calculations in the following: special for 1D Ising under RNG that renormalized partition function can be written like unrenormalized one new variables

$$x = e^{-4K}, \quad y = e^{-2h}, \quad w = e^{-4E}$$

and renormalized parameters in terms of those

$$x' = x(1+y)^2/(x+y)(1+xy)$$

$$y' = y(x+y)/(1+xy)$$

note: change in energy scale (w) does not impact behavior of free energy depending on x, y

Nelson/Fisher
Annals of Physics
91, 226 (1975)

consider $K>0$, ferromagnetic coupling

- line of paramagnetic fixed points
  $x^* = 1$, $T = \infty$
  all points starting with finite T end there
- fully ordered fixed point $x^* = 0$, $y^* = 0$
  at $T = 0$ and $H = \infty$

- ferromagnetic fixed point
  $x^* = 0 \quad , \; y^* = 1$ O

  ↳ Linearization around this fixed point

$$x' \sim 4x \quad , \quad \varepsilon = y - y^* = y - 1 \, , \; \varepsilon' \sim 2\varepsilon$$

hence Eigenvalues are $\lambda_x = 4, \; \lambda_\varepsilon = 2$

and $\quad y_x = 2 \quad , \; y_\varepsilon = 1$

free energy per spin

$$\bar{f}(x, \varepsilon) = b^{-1} \bar{f}(b^2 x, b\varepsilon)$$

by analytic continuation, b can be fixed at any value, even $b \neq 2$, hence we fix $b^2 x = 1$

$$\bar{f}(x, \varepsilon) = \sqrt{x}\, \bar{f}(1, \varepsilon/\sqrt{x})$$

if temperature t is replaced by $x = e^{-44}$, the following critical exponents are obtained

α	$3/2$
β	0
γ	$1/2$
δ	∞ discontinuity
ν	$1/2$
η	1