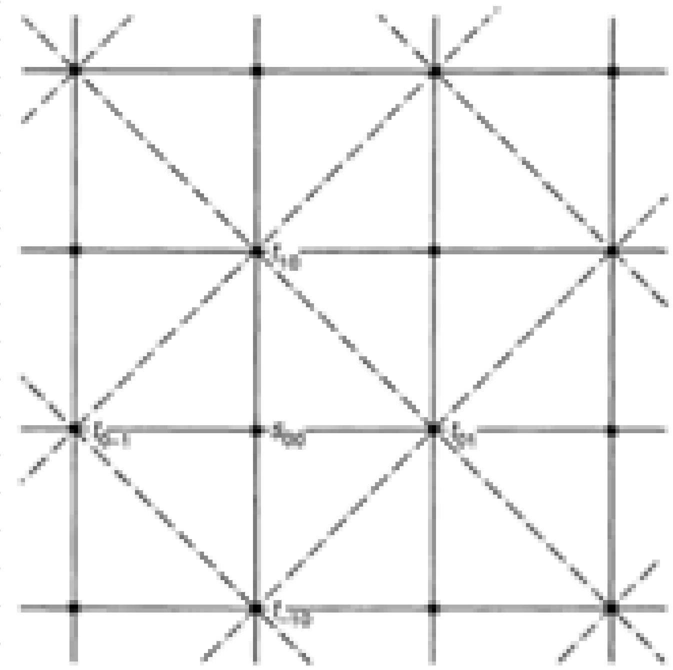
RNG in Higher Dimensions

decimation (= partial trace) of square lattice for 2D Ising model



$b = \sqrt{2}$

deleted spins s

kept spins t

reformulation of partition sum as product of single-s-spin terms

$$(*) \quad \exp\{K\, s_{00}(t_{01} + t_{0-1} + t_{10} + t_{-10})\}$$

the trace over s_{00} yields

$$2\cosh\{K(t_{01} + t_{0-1} + t_{10} + t_{-10})\}$$

rewriting that term using

$$a(K) = \ln 2 + (\ln\cosh 4K + 4\ln\cosh 2K)/8$$

$$b(K) = (\ln\cosh 4K)/8$$

$$c(K) = (\ln\cosh 4K - 4\ln\cosh 2K)/8$$

yields

$$\exp\{a(K) + b(K)[t_{-10}t_{01} + t_{01}t_{10} + t_{10}t_{0-1} + t_{0-1}t_{-10} + t_{-10}t_{10} + t_{0-1}t_{01}] + c(K)\, t_{-10}t_{01}t_{10}t_{0-1}\}$$

renormalized interactions involve
- nearest neighbours (as before)
- 2nd-nearest neighbours (new)
- four-spin terms (new)

and the renormalized Hamiltonian reads

$$\mathcal{H}' = \underbrace{2}_{\text{double counting 560 spins}} b(K) \sum_{\langle ij\rangle} t_i t_j + b(K) \sum_{[ij]} t_i t_j + c(K) \sum_{\text{square}} t_i t_j t_k t_l$$

$\langle ij\rangle$ nearest neighbors

$[ij]$ 2nd-nearest neighbours

problems:
- subsequent decimation impossible to be written as simple as (*)
- long-range interactions appear
- multi-spin couplings

resolution/approximation
- short-range interactions dominate critical behaviour
- truncate interactions at 2nd-nearest neighbour coupling

- keep only terms of order K^2 in expansions

with K, nearest-neighbour coupling and L, 2nd-nearest-neighbour coupling one gets the recursion

$$\boxed{\begin{aligned} K' &= 2K^2 + L \\ L' &= K^2 \end{aligned}}$$

terms like $t_{10} t_{01}$

↳ fixed points

$K^* = L^* = \infty$, ferromagnetic

$K^* = L^* = 0$, paramagnetic

and $\begin{pmatrix} K^* \\ L^* \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

↳ linearization: $\lambda_1 = 1.721$, $\lambda_2 = -0.387$

$$\nu = 1/y_1 = \frac{\ln b}{\ln \lambda_1} = 0.638 \quad (\text{not } 1)$$

only one relevant variable, only a single fixed point independent of K and L

↳ universality demonstrated by RNG

approximate results show nicely the topology of RNG flows, numbers of the exponents may be considerably off, and including higher terms may but is not guaranteed to yield better results (uncontrolled approximation!)

RNG in Momentum Space

ε - Expansion

real-space RNG: integrating out short length scales

RNG in momentum space: integrating out large momenta or wave vectors

Wilson/Kogut, Phys. Rep. 12, 75 (1974)

starting Hamiltonian (cf. Ginzburg-Landau)

$$\bar{\mathcal{H}} \sim \int dq\, (\tilde{r} + q^2)\, |m(q)|^2$$

$$+ \int dq_1 \int dq_2 \int dq_3\ \tilde{u}\ m(q_1)\, m(q_2)\, m(q_3) \times m(-q_1 - q_2 - q_3)$$

↳ first-order recursion equations (ε^1)

$$r' = 4\,[\,r + 3cu/(1+r)\,]$$
$$u' = 2^{\varepsilon}\,[\,u - 9cu/(1+r)^2\,]\,,\quad \varepsilon = 4-d$$

$d > 4$: stable fixed point at $r^* = u^* = 0$
↳ mean-field exponents

$d < 4$: u becomes extra relevant variable at mean-field fixed point, RNG flow driven away from $(r, u) = (0, 0)$
new fixed point at

$$\begin{pmatrix} r^* \\ u^* \end{pmatrix} = \frac{\varepsilon \ln 2}{9c} \begin{pmatrix} -4c \\ 1 \end{pmatrix}$$

RNG predicts upper critical dimension $d=4$

values for critical exponents, e.g. γ in 3D, $\varepsilon = 1$

ε^1 $\gamma = 1.17$

ε^2 $\gamma = 1.245$ best compared to $1.237..$

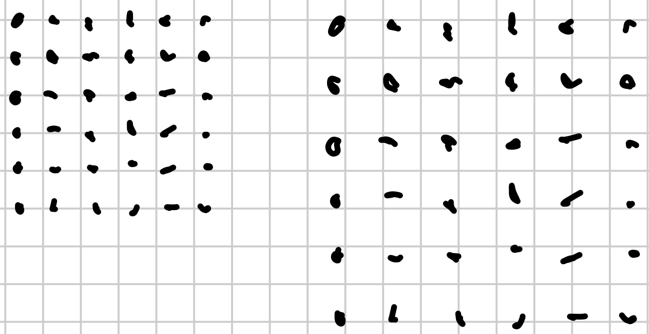
ε^3 $\gamma = 1.195$ (!)

Critical Exponents Ising Model

	$D=2$	$D=3$	mean-field
α	0	$0.11008..$	0
β	$1/8$	$0.326...$	$1/2$
γ	$7/4$	$1.237...$	1
δ	15	$4.789...$	3
ν	1	$0.629..$	$1/2$
η	$1/4$	0.032	0

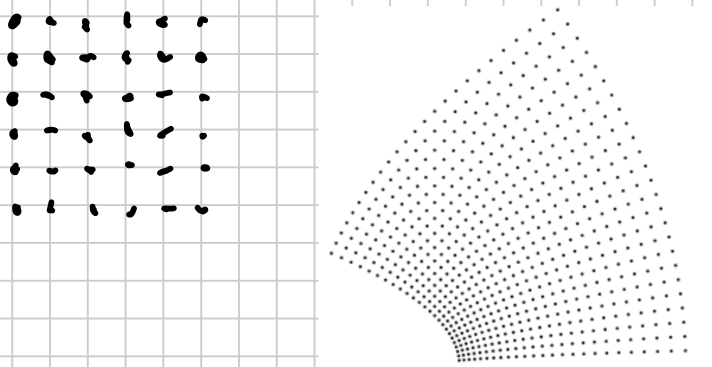
Scale invariance vs conformal invariance

scaling transformation with constant scaling factor b



in addition, original or transformed Hamiltonians may have rotational symmetries or other transformation invariances

conformal transformation with scaling factor depending continuously with position $b(\vec{r})$



locally angle-preserving mapping

transformation is locally (1) translation
(2) rotation
(3) dilation
but no shear!

in 2D: identification with complex plane
↳ holomorphic functions
Ising critical exponents are rational numbers