Understanding Traffic Flow

In the last ~20 years (theoretical) physicists have tried to apply methods from statistical physics to other areas, less obviously related to many-particle physics. Examples are

Econophysics: Using physical insights to understand how the economy and (financial) markets work

Socio-physics: Studying decision making in groups, voting systems, etc.

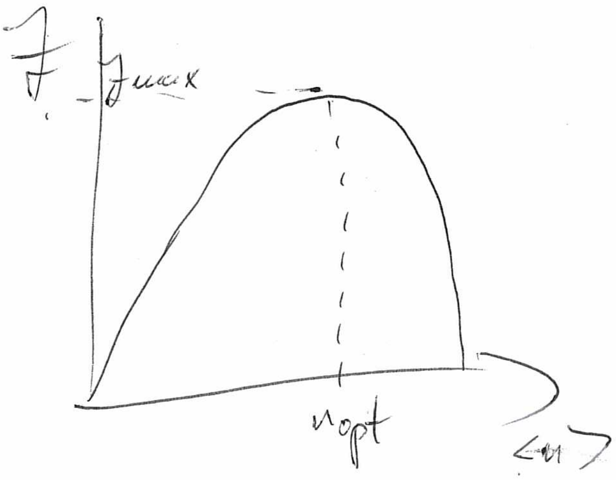
Biophysics: How many of the features found in living creatures can be described by "simple" physical laws? Which features are "genuinely biological"

It is not obvious that this is a usefull approach, after all:

- Market-participants tend to make complicated plans for how to trade or invest
- Human groups are composed of highly individualistic members
- Almost all forms of life down to bacteria have complicated signal-processing capabilities and do not interact via simple, deterministic interaction laws

Experience shows:
For the big picture we can often (but not always) forget about all the individual variation and trust in the averaging tendency of large numbers. Variation will then appear as noise

For infrastructure management it is interesting to know the fundamental diagram of traffic flow on a road, i.e. the relation between the vehicle density $\langle n \rangle$ and the vehicle flux $J = \langle vn \rangle$.
At low densities $\langle n \rangle \to 0$, we expect the typical speed to be independent of density such that $J \approx \langle v \rangle \langle n \rangle$
At very high densities we expect a traffic jam $J \to 0$. Somewhere in between the flux must thus be maximal



The Nagel-Schreckenberg-Model

A simple model for one-lane, one direction traffic

Cars sit on discrete sites $i = 1, \dots, N$ of a 1d lattice (with periodic boundary conditions). Each site can carry at most one car (no accidents)

At every timestep, each car has a velocity $v_i \in \{0, 1, \dots, v_{max}\}$

In every timestep the state of the system is updated through 4 steps

1. Acceleration

$$v_n \to \min(v_n + 1, v_{max})$$

2. Deceleration

Let d_n be the distance to the vehicle ahead

$$v_n \to \min(v_n, d_n - 1)$$

3. Randomization

If $v_n > 0$

$v_n \to \max(v_n - 1, 0)$ with probability $1-p$

4. Vehicle movement

$$x_n \to x_n + v_n$$

Simulations of this model show the spontaneous formation of traffic jams that move upstream in traffic.

With $v_{max} = 1$ and random sequential instead of parallel update steps, we arrive at the

<u>Asymmetric Simple Exclusion Process (ASEP)</u>

Consider a lattice $i = 1, \ldots, N$ with occupation numbers $\tau_i = 0, 1$. In every timestep Δt a particle at site i hops to site $i+1$ with propability Δt iff the site $i+1$ is empty. Particles enter at site 1 with probability α and exit on site N with probability β.

We are interested in the stationary state of this dynamics where the mean particle density $\langle \tau_i \rangle$ becomes time independent

We can calculate the mean density

$$\langle \tau_i \rangle = \sum_{\tau_1=0,1} \cdots \sum_{\tau_N=0,1} \tau_i P_N(\tau_1, \dots, \tau_N)$$

and correlation functions

$$\langle \tau_i \tau_j \rangle = \sum_{\tau_1=0,1} \cdots \sum_{\tau_N=0,1} \tau_i \tau_j P_N(\tau_1, \dots, \tau_N)$$

once we know the full N-site probability distribution $P_N(\tau_1, \dots, \tau_N)$.

We can write down a master equation

$$\frac{d}{dt} P_N = \sum_{\sigma_1} W^1_{\tau_1, \sigma_1} P_N(\sigma_1, \tau_2, \dots, \tau_N) + \sum_{\sigma_N} W^N_{\tau_N, \sigma_N} P_N(\tau_1, \dots, \tau_{N-1}, \sigma_N)$$
$$+ \sum_{i=1}^{N-1} \sum_{\sigma_i, \sigma_{i+1}} W_{\tau_i \tau_{i+1}, \sigma_i \sigma_{i+1}} P_N(\tau_1, \dots, \tau_{i-1}, \sigma_i, \sigma_{i+1}, \tau_{i+2}, \dots, \tau_N)$$

The only nonzero elements of the transition matrices are

$$W^1_{10} = -W^1_{00} = \alpha$$
$$W^N_{01} = -W^N_{11} = \beta$$
$$W_{01,10} = -W_{10,10} = 1$$

In the stationary state, we should be able to rewrite

$$\sum_{\sigma_i,\sigma_{i+1}} w_{\tau_i\tau_{i+1},\sigma_i\sigma_{i+1}} P_N(\tau_1,\dots,\sigma_i,\sigma_{i+1},\dots,\tau_N)$$

$$= -x_{\tau_i} P_{N-1}(\tau_1,\dots,\tau_{i-1},\tau_{i+1},\dots,\tau_N)$$
$$+ x_{\tau_{i+1}} P_{N-1}(\tau_1,\dots,\tau_i,\tau_{i+2},\dots,\tau_N)$$

with rates x_0, x_1 to be determined.
To make progress, we consider the unnormalized distribution $f_N(\tau_1,\dots,\tau_N)$ such that

$$P_N = f_N / Z_N$$

with

$$Z_N = \sum_{\tau_1=0,1} \cdots \sum_{\tau_N=0,1} f_N(\tau_1,\dots,\tau_N)$$

Following Derrida, we make the bold ansatz that f_N can be systematically calculated as the expectation value of a product of operators

$$f_N(\tau_1, \dots, \tau_N) = \langle W| \prod_{i=1}^{N} [\tau_i D + (1-\tau_i) E] |V\rangle$$

with an operator D for every occupied site, an operator E for every empty site, and two state vectors $\langle W|, |V\rangle$.
If we plug this ansatz into the master equation for the stationary state $\frac{d}{dt} P_N = 0$, we obtain conditions on the operators

$$DE = D + E$$

and on the state vectors

$$D|V\rangle = \frac{1}{\beta} |V\rangle$$

$$\langle W| E = \frac{1}{\alpha} \langle W|$$

With $C := D + E$ we then have

$$\langle \tau_i \rangle = \frac{\langle W| C^{i-1} D C^{N-i} |V\rangle}{\langle W| C^N |V\rangle}$$

and

$$\langle \tau_i \tau_j \rangle = \frac{\langle W| C^{i-1} D C^{j-i-1} D C^{N-j} |V\rangle}{\langle W| C^N |V\rangle}$$

If D and E commute, correlations cannot depend on the distance $j-i$ and must therefor vanish.
For commuting D and E we have

$$\langle W|V\rangle = \alpha\beta\langle W|ED|V\rangle = \alpha\beta\langle W|DE|V\rangle$$
$$= \alpha\beta\langle W|D+E|V\rangle = \alpha\beta\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\langle W|V\rangle$$

which implies $\alpha+\beta=1$. For all other parameters α, β, the operators D and E will not commute. Moreover, it can be shown that they can also not be represented by finite dimensional matrices.
From the algebraic properties alone, $\langle \tau_i\rangle$ and $\langle \tau_i \tau_j\rangle$ can be calculated exactly. Far from the boundaries one obtains the simple result

$$\langle \tau_i \rangle = \begin{cases} 1/2 & \\ \alpha & \text{for} \\ 1-\beta & \end{cases} \quad \begin{array}{l} \alpha, \beta > 1/2 \\ \alpha < 1/2, \alpha < \beta \\ \beta < 1/2, \beta < \alpha \end{array}$$

i.e, a constant mean density throught the bulk except for $\alpha = \beta < 1/2$ where

$$\langle \tau_n \rangle = \alpha + n(1-2\alpha)$$

linearly increasing along the chain.

The current J between two sites is even easier to calculate

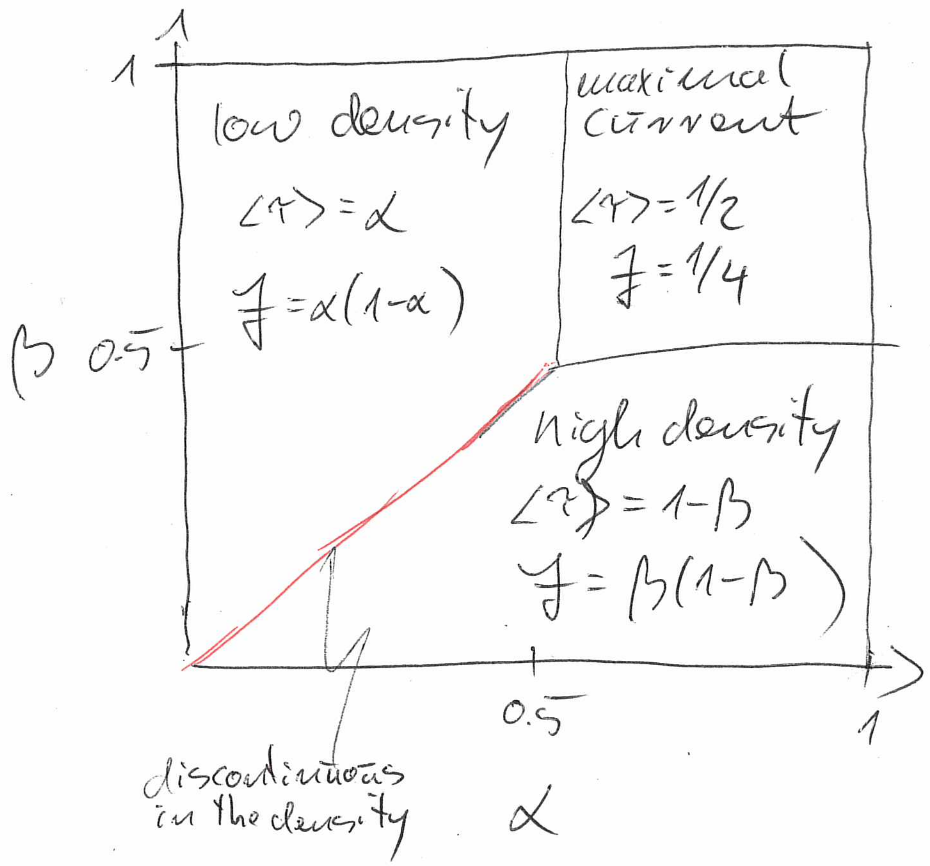
$$J = \langle \tau_i (1-\tau_{i+1}) \rangle = \frac{\langle W | C^{i-1} D E C^{N-i-1} | V \rangle}{\langle W | C^N | V \rangle}$$

$$= \frac{\langle W | C^{N-1} | V \rangle}{\langle W | C^N | V \rangle}$$

with the result that in the bulk

$$J = \begin{cases} 1/4 & \\ \alpha(1-\alpha) & \text{for} \\ \beta(1-\beta) & \end{cases} \quad \begin{array}{l} \alpha, \beta > 1/2 \\ \alpha < 1/2, \alpha < \beta \\ \beta < 1/2, \beta < \alpha \end{array}$$

This can be summarized in a phase diagram



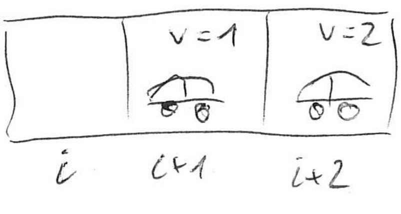
We thus obtain the fundamental relation

$$J(\langle\tau\rangle) = \langle\tau\rangle(1-\langle\tau\rangle)$$

symmetric around half-filling $\langle\tau\rangle = 1/2$

Note that we obtained a rich phase diagram for a one-dimensional but essentially non-equilibrium system

For simulations, the parallel update is the most natural approach while the random sequential update is amenable to analytical approaches. Switching from one to the other is not as innocuous as it may seem. For the random sequential update, all valid configurations can, in principle, be generated dynamically. Under the parallel update dynamics, however, there are so called Garden of Eden states that can only occur as initial conditions but can never be reached dynamically. E.g. the following configuration in the Nagel-Schreckenberg model



is perfectly valid as an initial condition but cannot be dynamically created as at time t-1 both cars would have come from site i, which is not a valid configuration. It is a nontrivial question how important the garden of eden states are