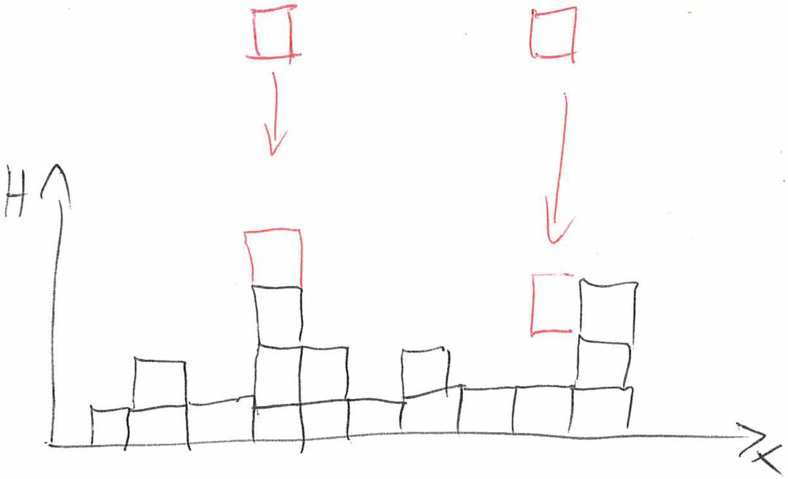
A possible generalization is to go from the totally asymmetric process described above (also called TASEP) to a process where the particles hop with a propability p to the right and with a probability q to the left.
We will see below that this model can be used to investigate surface growth

Surface Growth

Imagine we have a (one-dimensional) surface, discretized into lattice sites and we rain down building blocks on that surface that fall straight down until they touch an existing block



We assume that the falling blocks start at uncorrelated random times and positions.

We are interested in the height profile $H(x,t)$ at some time t, generated by this ballistic deposition model

Qualitatively we can argue that this model will produce surfaces that are not "too rough":

(1) Deep valleys with a high curvature of the surface will quickly be filled in

(2) strong gradients will also be

smoothed out by blocks attaching to the sides

A coarse-grained, continuum equation for the height-field that respects these features is given by the famous

Kardar - Parisi - Zhang - Equation

$$\partial_t H = \nu \partial_x^2 H + \frac{1}{2}\lambda(\partial_x H)^2 + \sqrt{D}\,\xi$$

commonly abbreviated KPZ-equation. Here, ν parametrizes the curvature driven growth. Gradients driving growth is parametrized by λ and the random raining down is modelled by the white noise $\xi(x,t)$

$$\langle \xi(x,t)\,\xi(x',t')\rangle = \delta(x-x')\,\delta(t-t')$$

and the "diffusion constant" D controlls the driving strength

Formally, this is a nonlinear stochastic partial differential equation.
Unfortunately, it does not make sense mathematically. In essence, the $(\partial_x H)^2$-term will almost surely be infinite.
By rescaling the time, x-position and height conviently, we can get rid of all parameters and study

$$\partial_t H = \frac{1}{2}\partial_x^2 H - \frac{1}{2}(\partial_x H)^2 + \xi$$

To this we can apply the Hopf-Cole-Transformation: If we forget about the stochastic nature of ξ for a moment we define

$$Z(x,t) := e^{-H(x,t)}$$

or

$$H(x,t) = -\ln Z(x,t)$$

We then have

$$\frac{\partial_t Z}{Z} = -\frac{1}{2}\,\frac{Z\partial_x^2 Z - (\partial_x Z)^2}{Z^2} - \frac{1}{2}\frac{(\partial_x Z)^2}{Z^2} + \xi$$

or

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z - Z\xi$$

Without the white noise term, this is just the form of the heat-equation so that the equation above is also called the stochastic heat equation

Although it contains multiplicative noise $Z\xi$, it is mathematically well-defined and well behaved

If we smooth out the solutions $Z(x,t)$ over a small length-scale κ, the logarithm of Z is well defined and we obtain a modified KPZ equation

$$\partial_t H = \frac{1}{2}\partial_x^2 H - \frac{1}{2}\left[(\partial_x H)^2 - \frac{1}{2\sqrt{\pi}\,\kappa}\right] + \xi$$

which now makes sense mathematically at least for finite κ.

It is tempting to generalize the KPZ-equation to higher dimensional surfaces

$$\partial_t H(\underline{x},t) = \frac{1}{2}\nabla^2 H(\underline{x},t) - \frac{1}{2}|\nabla H(\underline{x},t)|^2 + \xi(\underline{x},t)$$

but it is not known yet how to make mathematical sense of this as the Hopf-Cole trick does not work any more.

Let's assume that for $\varepsilon \to 0$, the height profile admitts a scaling form

$$H(x,t) = \varepsilon^{\beta} H\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^{z}}\right)$$

with two scaling exponents β and z

without proof, I will claim that $\beta = 1/2$

The variance of the white noise

$$\langle \xi(x,t)\xi(x',t')\rangle = \delta(x-x')\delta(t-t')$$

has dimensions of inverse length times inverse time such that ξ should scale like

$$\xi(x,t) = \varepsilon^{\frac{z+1}{2}}\xi(\varepsilon x, \varepsilon^z t)$$

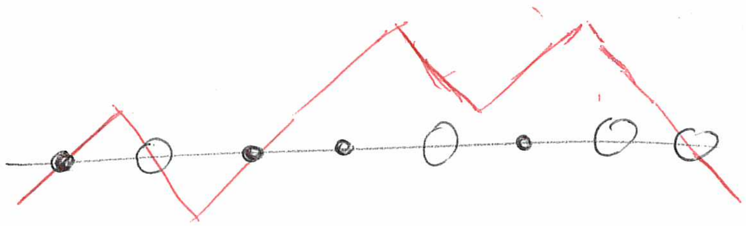
If we plug this into the KPZ-equation we find

$$\partial_t H = \frac{1}{2}\varepsilon^{2-z}\partial_x^2 H - \frac{1}{2}\varepsilon^{\frac{3}{2}-z}(\partial_x H)^2 + \varepsilon^{1-\frac{1}{2}z}\xi$$

For the nonlinear term to stay relevant as $\varepsilon \to 0$, we need $z = 3/2$

It is believed that $\beta = 1/2$, $z = 3/2$ characterises a universality class of (growth) processes. I.e. various microscopic models will all be

Interestingly, this can be mapped onto the ASEP model.
Identify, say, a positive slope with an occupied site and a negative slope with an empty site

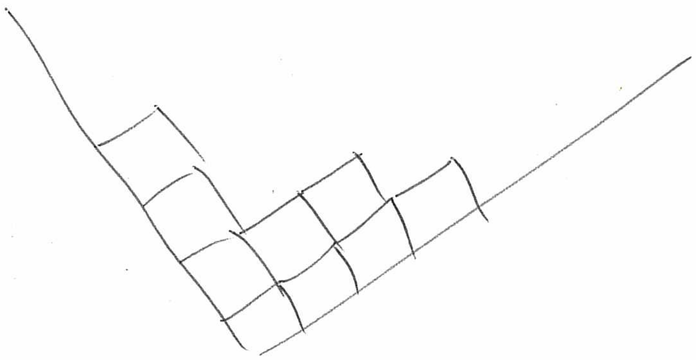


A particle hopping to the right

is precisely one of the steps in the corner growth model and the same holds obviously for the reverse process. This means the coarse-grained particle density $u(x,t)$ of the ASEP should follow described by the KPZ-equation on macroscopic length and time scales. One of those is the

Corner Growth Model

Given a configuration of tilted blocks in a wedge



we turn valleys ∨ into hills ◇ with a rate q and hills ◇ into valleys ∨ with a rate p such that we obtain a positive growth rate $\gamma = q - p > 0$

as the gradients of the height profile $\nabla H(x,t)$. If we take the gradient of the KPZ-equation

$$\partial_x \partial_t H = \frac{1}{2}\partial_x \partial_x^2 H - \frac{1}{2}\partial_x(\partial_x H)^2$$
$$= \frac{1}{2}\partial_x^2 \partial_x H - (\partial_x H)(\partial_x^2 H)$$

we obtain the Bürger's equation

$$\partial_t u + u\partial_x u = \frac{1}{2}\partial_x^2 u$$

where we neglected the fluctuations for simplicity.

A more carefull derivation shows that the term on the right hand side acquires a factor ε and thus vanishes on the longest lengthscales. There, the gradients in the height, or the particle density, respectively follow from the

Inviscid Bürgers' Equation

$$\partial_t u + u \partial_x u = 0$$

The Bürgers' Equation was originally obtained from the Navier-Stokes-Equation for an incompressible fluid

$$D_t u(\underline{x},t) = -\nabla p(\underline{x},t) + \eta \nabla^2 u(\underline{x},t)$$

If we assume a constant pressure field and concentrate on just one-dimensional perturbations we obtain

$$\partial_t u + u \partial_x u = \eta \partial_x^2 u$$

which explains the name inviscid Bürgers' equation as that is formally obtained by setting the viscosity η to zero

While the link between the ASEP and 1d fluid flow may be natural, the link between fluid flow and surface growth is quite surprising.

We can solve the nonlinear partial differential Burgers' equation by the Method of Characteristics

The solution $x(t)$ of the ordinary differential equation

$$\frac{dx}{dt} = u(x,t)$$

is called a characteristic (curve)

The nice property of these is that $u(x,t)$ is constant along a characteristic

$$\frac{d}{dt} u(x(t),t) = \frac{\partial u}{\partial t}(x(t),t) + \frac{dx}{dt}\frac{\partial u}{\partial x}(x(t),t)$$

$$= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

This implies that characteristics must be straight lines

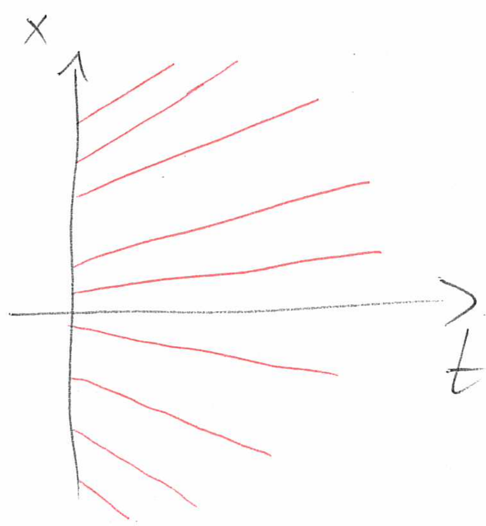
$$x(t) = u_0 t + x_0$$

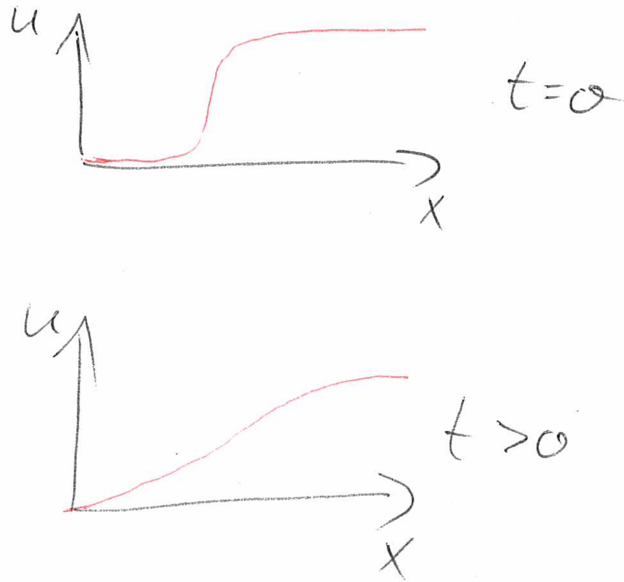
fully characterised by the initial data x_0, u_0

A spatial distribution $u_0(x)$ at time $t=0$ will thus evolve into a velocity field at time $t>0$ given implicitly by

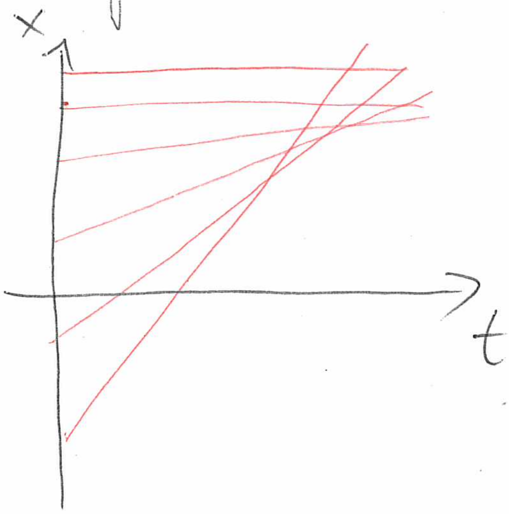
$$u(x,t) = u_0\big(x - t\,u(x,t)\big)$$

Except for the boring case $u_0(x) \equiv u_0$ one has either Rarefaction Waves

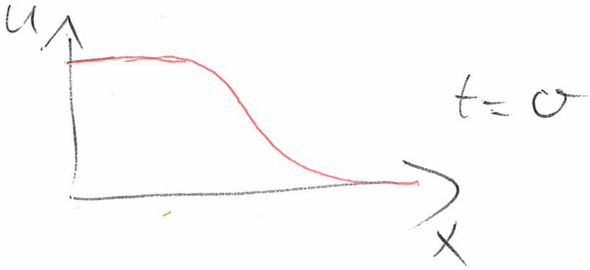


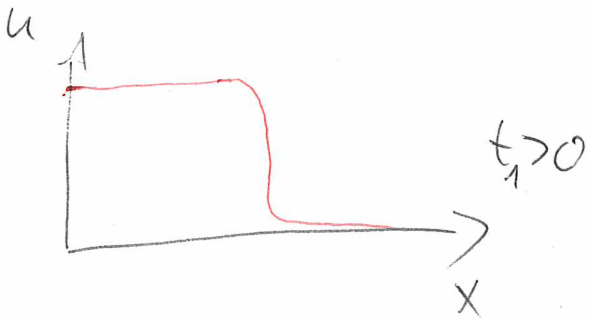


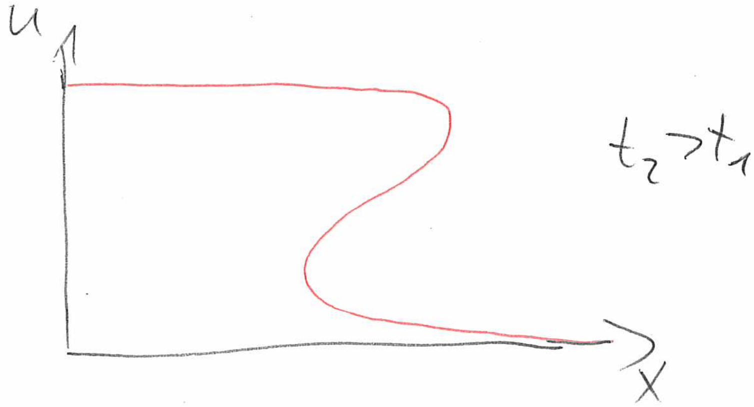
or a problem



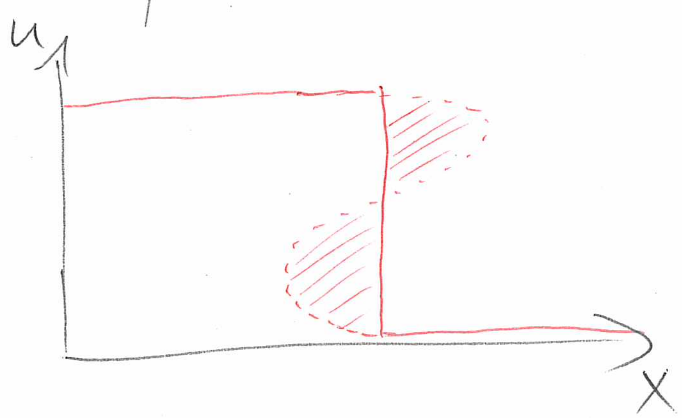
After some critical time t^* the characteristics cross. The velocity $u(x,t)$ is given by the slope of the characteristic at that point. When the characteristics cross, the velocity seems to become multi-valued







Instead one finds that the multi-valued part should be replaced by a vertical line where mass conservation dictates an equal-area-rule



Such an discontinuous jump in the flow velocity is called a <u>shock wave</u>. Shock waves in the ASEP (where they manifest as discontinuities in the density) have been studied with the algebraic methods introduced above.