Prof. Dr. M. Sperl
Dr. T. Kranz

3. Problem Set
Advanced Statistical Physics

Due Date: Thursday, November 10, 10am

Please indicate your name and the number of your group on the first page!

Problem 7 *Mean-Field-Approximation for a Lattice Gas* **12 Punkte**

A fluid shall be modeled by a variable number N of particles occupying a lattice with M sites. The short range *hard core* repulsion of the particles shall be taken into account by demanding that each lattice site can be occupied by at most one particle (occupation number $n_i \in \{0, 1\}$ for $i = 1, 2, \ldots, M$ and $N = \sum_{i=1}^{M} n_i$). The attraction at longer distances shall be modeled by a box potential between neighboring particles. Here we consider a mean-field model with an average potential depth ε/M (and $\varepsilon > 0$) for all pairs of lattice sites, i.e.,

$$\mathcal{H} = -\frac{\varepsilon}{2M} \sum_{i,j=1}^{M} n_i n_j$$

a) In order to calculate the grand canonical partition function $\mathcal{Z} = \sum_{\{n\}} e^{-\beta\mathcal{H}+\beta\mu N}$, decouple the interactions with a Hubbard-Stratonovich-Transformation, introducing an auxiliary field ν.

b) Consider the thermodynamic limit $M \to \infty$ and perform a saddle-point approximation. Determine the (grand canonical) function $g(\nu)$ and an equation for $\overline{\nu}$, dominating the integral.

c) Determine the relationship between $\overline{\nu}$ and the average particle density $\varrho := \langle N\rangle/M$. Derive the self-consistency equation for ϱ.

Problem 8 *The q-state Potts Model* **10 Punkte**

The Potts model is defined by the following Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij\rangle} \delta_{\sigma_i\sigma_j} - H \sum_i \delta_{1,\sigma_i}$$

with $\sigma_i \in \{1, 2, \ldots, q\}$ and $\delta_{\alpha\beta}$ the Kronecker symbol. Consider a one-dimensional chain $i = 1, \ldots, N$ with periodic boundary conditions

a) Assume $H \equiv 0$ and $q = 3$. Calculate the free energy with the transfer matrix method. To this end define a suitable transfer matrix T. To caluclate its eigenvalues λ it may be helpful to plot the characteristic polynomial as a function of $x := e^{\beta J} - \lambda$. Justify whether you need to know the eigen vectors.

b) Now reintroduce the field $H \neq 0$ and consider $q = 2$. Use another transfer matrix to calculate the free energy F

Problem 9 *Duality: The next order* **8 Punkte**

Consider the Ising model on a square lattice with N sites.

a) Draw all possible configurations for the third excitation level of the low temperature expansion. Argue how many of them can occur.

b) Draw all third order graphs for the high temperature expansion and justify their multiplicity.

c) Explicitly show the duality relation by drawing the corresponding graphs from a) and b) on top of each other and verifying that the multiplicities match.