Prof. Dr. M. Sperl
Dr. T. Kranz

6. Problem Set
Advanced Statistical Physics

Due Date: Thursday, December 1, 10am

Please indicate your name and the number of your group on the first page!

Problem 16 *One more Critical Exponent* **12 Punkte**

Consider the generic Landau free energy

$$\mathcal{F}[\phi] = f_0 V + \int d^d x \left[At\phi^2(\boldsymbol{x}) + \frac{\kappa}{2}\big(\nabla\phi(\boldsymbol{x})\big)^2 \right]$$

valid above T_c. Here V denotes the volume of the system

a) Rewrite $\mathcal{F}$ in the Fourier domain and show that the free energy is given as

$$F = f_0 V - \frac{1}{2} k_B T \sum_{\substack{\boldsymbol{k} \\ k<\Lambda}} \ln \frac{2\pi V k_B T}{2At + \kappa k^2}$$

where Λ^{-1} is the coarse graining length scale.
Hint: Gaussian integrals!

b) Calculate the heat capacity $c_V = -T\frac{\partial^2 (F/V)}{\partial T^2}$ and explicitly evaluate the resulting sums over $\boldsymbol{k}$. To this end, turn them into integrals and change the integration variable to $\boldsymbol{q} := \xi \boldsymbol{k}$ with the correlation length ξ. Discuss the behavior of the integrands for $q \to 0$ and $q \to \infty$ depending on the dimension of space. Find the *most divergent* term and show that $c_V \sim t^{-\alpha}$ with yet another critical exponent

$$\alpha = \begin{cases} 2 - d/2 & \text{for } d < 4 \\ 0 & \text{for } d > 4 \end{cases}$$

Problem 17 *Scaling- and Hyperscaling Relations* **6 Punkte**

The fixed point condition for the free energy density in the renormalization procedure reads (cf. the lecture)

$$f(t, h) = b^{-d} f(tb^{y_t}, hb^{y_h})$$

with an arbitrary scale factor b

a) Choose $b = t^{-1/y_t}$ to show that the critical exponent $\alpha = 2 - d/y_t$ for the divergence of the heat capacity $c_V \sim t^{-\alpha}$

b) Use the scaling form of the free energy density derived in a) to calculate the scaling forms of the magnetization $m \sim \partial f/\partial h$ and the susceptibility $\chi \sim \partial m/\partial h$. Eliminate y_h from the two equations to prove the *scaling relation*

$$\alpha + 2\beta + \gamma = 2$$

c) Argue that the free energy *density* at zero field should diverge like $f(t) \sim \xi^{-d}$ to derive the *hyperscaling relation*

$$2 - \alpha = \nu d$$

valid for $d < 4$.

Problem 18 *Renormalization by Decimation* **12 Punkte**

Consider a 1d Ising chain with Hamiltonian

$$\beta\mathcal{H} = -K \sum_i s_i s_{i+1} - E \sum_i 1$$

where we added a constant energy term E for later convenience. The partition function is then given as

$$Z = \sum_{\{s_i = \pm 1\}} \prod_i e^{K s_i s_{i+1} + E}$$

a) Implement a decimation procedure where in each step every other spin is removed. To this end consider a spin s_j to be eliminated and perform the sum over $s_j = \pm 1$ in the partition function. We assume that this procedure should result in a Ising chain with half the number of spins and thus the contribution from the eliminated spin s_j results from the interactions of its neighbors. I.e., we assume it to be of the form

$$e^{K' s_{j-1} s_{j+1} + E'}$$

with a new coupling constant K' and a new energy E'. By considering all possible values of s_{j-1}, s_{j+1} determine two equations for the two unknowns K', E'.

b) Use $x := e^{-4K}$ to determine the recursion relation $x \to x' = \mathcal{R}x := x'(x)$. Show graphically, that this renormalization transformation has two fixed points, only one of which is stable. How does this relate to what you know about the phase behavior of the 1d Ising model?

c) Linearize the renormalization transformation $\mathcal{R}$ around the low-temperature fixed point x_0 to first order in $\delta x := x - x_0$. From dimensional analysis we know that the correlation length after ℓ decimation steps is related to the original correlation length by

$$\xi(\mathcal{R}^\ell x) = 2^{-\ell} \xi(x)$$

For x close to x_0, use the linearized transformation to show that the correlation length diverges for $T \to 0$ as

$$\xi \sim e^{2K} \quad \text{for } K \to \infty$$