Prof. Dr. M. Sperl
Dr. T. Kranz

WiSe 2016/17

7. Problem Set
Advanced Statistical Physics

Due Date: Thursday, December 8, 10am

Please indicate your name and the number of your group on the first page!

Problem 19 *Birds and Budworms* **6 Punkte**

Spruce budworms proliferate on a characteristic time scale τ until they have reached the carrying capacity C of their tree. Birds like budworms but also eat other insects if there are not enough budworms. Even if they concentrate on the budworms, they can only eat them at a maximum rate ω_0. A idealized model to capture the time evolution of the number of budworms $n(t)$ is given by

$$\tau \frac{dn}{dt} = n(1 - n/C) - \omega_0 \tau \frac{n^2}{1 + n^2}.$$

Close to the stationary state $dn/dt = 0$, it is reasonable to assume that the time evolution is driven by the gradients in a generalized free energy functional, $dn/dt \sim -d\mathcal{F}/dn$.

a) Determine the free energy functional $\mathcal{F}$

b) Plot $\mathcal{F}$ to determine the phase behavior close to the critical point $C^c = 3\sqrt{3}$, $\omega_0^c \tau = 8/3\sqrt{3}$ as the eating rate $\omega_0 \tau$ varies. Discuss your findings in terms of the bird-budworm ecosystem.

Problem 20 *More on the Gaussian Model* **12 Punkte**

In Problem 16 you have seen that we can calculate the exact free energy if the Landau free energy is quadratic in the order parameter. Here we will look at the Gaussian model again and renormalize it.

$$\beta \mathcal{F}_0[\phi] = \int d^d x \left[A t \phi^2(\boldsymbol{x}) + \frac{\kappa}{2} \big(\nabla \phi(\boldsymbol{x})\big)^2 \right]$$

a) Calculate the Fourier transform $G_0(k)$ of the correlation function $G_0(|\boldsymbol{x}_1 - \boldsymbol{x}_2|) := \langle \phi(\boldsymbol{x}_1) \phi(\boldsymbol{x}_2) \rangle$.

b) Split the order parameter field in to *slow modes* $\phi_s(\boldsymbol{k})$ and *fast modes* $\phi_f(\boldsymbol{k})$ such that

$$\phi(\boldsymbol{k}) = \begin{cases} \phi_s(\boldsymbol{k}) & \text{for } k < \Lambda/b \\ \phi_f(\boldsymbol{k}) & \text{for } \Lambda/b < k < \Lambda \end{cases}$$

where Λ is the short distance cutoff and $b > 1$ the scale factor. Write $\mathcal{F}_0[\phi] = \mathcal{F}_s[\phi_s] + \mathcal{F}_f[\phi_f]$ such that the partition function

$$Z_0 = \int \mathcal{D}\phi e^{-\beta \mathcal{F}_0[\phi]} = Z_0^f \int \mathcal{D}\phi_s e^{-\beta \mathcal{F}_s[\phi_s]}$$

with the obvious definition of the fast part of the partition function, Z_0^f. Instead of decimating spins, we would like to integrate out the fast modes (i.e., we drop Z_0^f and renormalize $\mathcal{F}_s \to \mathcal{F}_0$).

To this end we rescale the wave vectors $\boldsymbol{k}' = b\boldsymbol{k}$ and introduce a rescaled temperature t', and a rescaled stiffness κ' such that $\mathcal{F}_s[\phi_s(\boldsymbol{k}'/b)] \equiv \mathcal{F}_0[\phi'(\boldsymbol{k}')]$ where $\phi'(\boldsymbol{k}') = \zeta^{-1}\phi_s(\boldsymbol{k}'/b)$ is the rescaled field. Determine ζ from the requirement that the critical ($t = 0$) fluctuations $G_0(k)$ remain unchanged under renormalization. Determine the renormalization transformations $\kappa'(\kappa)$ and $t'(t)$.

c) Discuss the fixed points of the renormalization transformation $t \to t'$.

Problem 21 *The ϕ^4-Model* **12 Punkte**

Let us add a fourth order term to the Gaussian model

$$\beta\mathcal{F}[\phi] = \beta\mathcal{F}_0[\phi] + \beta u\mathcal{F}_1[\phi]$$

where $\beta\mathcal{F}_1[\phi] = \int d^dx\phi^4(\boldsymbol{x})$ and $u \ll 1$ is a small parameter.
Hint: It's probably a good idea to do Problem 20 first.

a) Convince yourself that we can write the partition function as

$$Z = Z_0^f \int \mathcal{D}\phi_s e^{-\beta\mathcal{F}_s[\phi_s]}\langle e^{-\beta u\mathcal{F}_1[\phi]}\rangle_f$$

where the fast average is defined as

$$\langle A\rangle_f := \frac{1}{Z_0^f}\int \mathcal{D}\phi_f A e^{-\beta\mathcal{F}_f[\phi_f]}.$$

Show that

$$\ln\langle e^{-\beta u\mathcal{F}_1[\phi]}\rangle_f = -u\beta\langle\mathcal{F}_1[\phi]\rangle_f + \mathcal{O}(u^2).$$

b) Write $\beta\mathcal{F}_1$ in the Fourier domain in terms of ϕ_s and ϕ_f and argue why only a term with no fast modes, $\mathcal{F}_1^{(0)}$, and a term with two fast and two slow modes, $\mathcal{F}_1^{(2)}$, will be important if we are going to integrate out the fast modes in the end. To this end show that $\beta\langle\mathcal{F}_1^{(0)}\rangle_f = \beta\mathcal{F}_1^{(0)}$ and

$$\beta\langle\mathcal{F}_1^{(2)}\rangle = 6\int_{\Lambda/b}^{\Lambda}\frac{d^dk}{(2\pi)^d}G_0(k)\int_0^{\Lambda/b}\frac{d^dp}{(2\pi)^d}|\phi_s(\boldsymbol{p})|^2$$

c) Convince yourself that to lowest order in u we can now write the partition function as

$$Z = Z_0^f \int \mathcal{D}\phi_s e^{-\beta\mathcal{F}_s[\phi_s]+\beta u\mathcal{F}_1^{(0)}[\phi_s]+\beta u\langle\mathcal{F}_1^{(2)}\rangle[\phi_s]}$$

Write down the modified renormalization transformation for t

d) What is the renormalization transformation for u and what does it imply for our perturbative approach?

Studentische Vollversammlung der Physik (Student Assembly) Thursday December 8, 12pm in HS III. Organized by the *Fachschaft* you will get all the information about the upcoming elections: What are the elections about? Who can you vote for? Why is it important? In addition there will be elections of the representatives for some committees within the physics department.

Studentische Wahlen (University Elections) taking place in the week December 12 – 16