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WiSe 2016/17

8. Problem Set
Advanced Statistical Physics

Due Date: Thursday, December 15, 10am

Please indicate your name and the number of your group on the first page!

Problem 22 *Random Walk* **10 Punkte**

Consider a particle starting at the origin that jumps around in space

$$\boldsymbol{r} \to \boldsymbol{r} + \boldsymbol{a}$$

where the jumps take place at discrete time $t = N\Delta t$ and the jumps follow a probability distribution $p(\boldsymbol{a})$ with zero mean and a finite variance $\langle a^2 \rangle < \infty$. The probability distribution to find the particle at position $\boldsymbol{x}$ after N steps shall be denoted by $P_N(\boldsymbol{x})$

a) Focus on times $t \gg \Delta t$ and positions sufficiently far away form the origin such that $x^2 \gg \langle a^2 \rangle$. Write down the recursion equation that relates P_N to P_{N-1} and approximate it under the assumptions given above to show that we can promote the discrete $P_N(\boldsymbol{x})$ to a probability density $\varrho(\boldsymbol{x}, t)$ that is a solution of the diffusion equation

$$\frac{\partial}{\partial t}\varrho(\boldsymbol{x}, t) - D\nabla^2 \varrho(\boldsymbol{x}, t) = 0$$

What determines the diffusion constant D?

b) Solve the diffusion equation by transforming to Fourier space and back again for the initial conditions

   i. $\varrho(\boldsymbol{x}, 0) = n_0 \delta(\boldsymbol{x})$

   ii. In one dimension $\varrho(x, 0) = n_0 \,\mathrm{rect}_\ell(x)$ where

$$\mathrm{rect}_\ell(x) := \begin{cases} 1/\ell & \text{for } |x| < \ell/2 \\ 0 & \text{otherwise} \end{cases}$$

Problem 23 *Yvon Theorems* **8 Punkte**

Several theorems in kinetic theory are connected to the name of Yvon. Two of them are given below

a) Show that for any function f that only depends on the positions of particles

$$\left\langle \frac{\partial f(\{\boldsymbol{r}_j\})}{\partial \boldsymbol{r}_i} \right\rangle = \beta \left\langle \frac{\partial U}{\partial \boldsymbol{r}_i} f(\{\boldsymbol{r}_j\}) \right\rangle$$

where $U = U(\{\boldsymbol{r}_j\})$ is the interaction potential of the particles

b) Show the following relation between the Liouville operator $\mathcal{L}$ and the Poisson bracket $\{\cdot, \cdot\}$

$$\langle X | i\mathcal{L} Y \rangle = -k_B T \langle \{X^*, Y\} \rangle$$

Problem 24 *A rarefied gas* **12 Punkte**

The pair-correlation function of a fluid with mean diensity n

$$ng(\boldsymbol{r}) = \frac{1}{N}\left\langle \sum_{i\neq j} \delta(\boldsymbol{r}+\boldsymbol{r}_i-\boldsymbol{r}_j) \right\rangle$$

indicates the probability to find pair of particles separated by $\boldsymbol{r}$. In an isotropic fluid $g(\boldsymbol{r}) = g(r)$. The structure factor is related to $g(r)$ by a Fourier transform

$$S(q) = 1 + n\int d^3r[g(r)-1]e^{i\boldsymbol{q}\cdot\boldsymbol{r}}$$

a) Consider a rarefied gas of hard spheres of diameter d. I.e., the interaction potential is given by

$$\beta U(r) = \begin{cases} \infty & \text{for } r < d \\ 0 & \text{for } r > d \end{cases}$$

Determine the structure factor using the *mean spherical approximation* $g(r) \approx \exp[-\beta U(r)]$. Plot $S(qd)$ for a volume fraction $\varphi := \pi n d^3/6 = 0.05$.

b) Determine the compressibility κ_T from the structure factor and write the correction term to the ideal gas compressibility in terms of the volume v of a hard sphere.

c) Use κ_T to determine the thermal equation of state $p(n,T)$. Expand the pressure p to second order in the density to verify the form

$$p = nk_BT\left[1 + B(T)n\right] + \mathcal{O}(n^3).$$

Explicitly determine the *second virial coefficient* $B(T)$ for the rarefied hard sphere gas.

Studentische Vollversammlung der Physik (Student Assembly) Thursday December 8, 12pm in HS III. Organized by the *Fachschaft* you will get all the information about the upcoming elections: What are the elections about? Who can you vote for? Why is it important? In addition there will be elections of the representatives for some committees within the physics department.

Studentische Wahlen (University Elections) taking place in the week December 12 – 16