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WiSe 2016/17

9. Problem Set
Advanced Statistical Physics

Due Date: Thursday, December 22, 10am

Please indicate your name and the number of your group on the first page!

Problem 25 *Generalized Harmonic Oscillator* **10 Punkte**

Consider the equation of motion (EOM)

$$\ddot{S}(t) + \nu\dot{S}(t) + \Omega^2 S(t) + \int_0^t d\tau M(t-\tau)\dot{S}(\tau) = 0$$

together with the initial conditions $S(0) = S > 0$ and $\dot{S}(0) = 0$ and $\nu, \Omega > 0$. Moreover $M(t)$ is assumed to be an equilibrium correlation function.

a) Perform a Laplace-Transformation of the EOM and solve it for $\hat{S}(z)$.
Note: Remember the convention from the lecture: $\hat{S}(z) = i\int_0^\infty dte^{izt}S(t)$.

b) Derive a short-time expansion for $S(t)$ up to and including terms of $\mathcal{O}(t^4)$. Calculate the high-frequency expansion of $\hat{S}(z)$ up to and including terms of $\mathcal{O}(z^{-5})$. Compare the two expansions.

Problem 26 *Model of a Rubber Band, Part II* **10 Punkte**

In problem 3 we introduced a simple model for a rubber band. Here we will extend the analysis of this model

a) Calculate the necessary (entropic) force $f = -T\frac{\partial S}{\partial L}$ to keep the chain at a prescribed length L. Determine the force-extension relation $L(f, T)$. What are the conditions for Hooke's law to apply?

b) Let us introduce an external field that biases the chain in one direction: Assume every segment pointing to the left costs an energy $\epsilon > 0$. Determine the entropy per segement S/N as a function of the specific energy $E/(N\epsilon)$.

c) Calculate and plot the temperature of the rubber band as a function of ϵ. Discuss the different regimes.

Problem 27 *The Virial Series* **10 Punkte**

Writing the equation of state

$$\frac{p}{k_B T} = n + B_2 n^2 + B_3 n^3 + \dots$$

as a Taylor series in the density $n = N/V$ is called a *Virial Expansion* and the expectation is that the higher order terms can be neglected at low densities.

a) Calculate the chemical potential $\mu(N, V, T)$ of an ideal gas and show that $\mu = \mu(n, T)$. Show that the *fugacity* $z := e^{\beta\mu}$ vanishes as $n \to 0$.

b) Using the *canonical* partition function $Z(N) \equiv Z(T, V, N)$ we introduce the *grand canonical* partition function

$$\mathcal{Z} = \sum_{N=0}^{\infty} Z(N) z^N .$$

Using the grand canonical partition function, expand the density $n(z) = a_1 z + a_2 z^2 + \mathcal{O}(z^3)$ to second order in the fugacity z. Invert $n(z)$ to obtain $z(n)$ to second order in the density n and use the grand canonical potential $G = k_B T \ln \mathcal{Z}$ to show that the second virial coefficient

$$B_2 = \frac{V}{2} \left(1 - \frac{2Z(2)}{Z^2(1)} \right)$$

can be written in terms of the one- and two-particle canonical partition functions.
Hint: Remember $G = PV$.

c) Consider a system of particles with a spherically symmetric interaction potential $U(r)$ and compute $Z(1), Z(2)$ to show that

$$B_2 = -2\pi \int_0^{\infty} dr r^2 \left(e^{-\beta U(r)} - 1 \right)$$