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WiSe 2016/17

11. Problem Set
Advanced Statistical Physics

Due Date: Thursday, January 19, 10am

Please indicate your name and the number of your group on the first page!

Problem 31 *Langevin-Equation for a Particle in a Harmonic Trap* **8 Punkte**

A small colloidal particle with mass m shall be trapped in a harmonic potential $U = m\omega_0^2 x^2/2$. The equation of motion for its (one-dimensional) position $x(t)$ and velocity $v(t)$ shall be given by the *Langevin-Equation*

$$\dot{x}(t) = v(t)$$
$$\dot{v}(t) = -\gamma v(t) - \omega_0^2 x(t) + f(t)/m.$$

Here $\gamma > 0$ is the friction coefficient and $f(t)$ the force exerted by the solvent molecules. This random force shall have zero mean $\langle f(t)\rangle = 0$ and be δ-correlated in time $\langle f(t_1)f(t_2)\rangle = m^2\Gamma\delta(t_1 - t_2)$ with the strength parametrized by Γ. Let the initial conditions be $x(t = 0) = x_0$ and $v(t = 0) = v_0$.

a) Solve the Langevin equation for one realization of the random force $f(t)$

b) If the particle is in thermal equilibrium with the solvent, energy-equipartition

$$\frac{m}{2}\omega_0^2\langle x_0^2\rangle_H = \frac{m}{2}\langle v_0^2\rangle_H = \frac{1}{2}k_B T$$

holds. Here $\langle\cdot\rangle_H$ is the canonical average over the intial conditions. Assume vanishing correlations between positions and velocities, $\langle x_0 v_0\rangle_H = 0$. Calculate the velocity-autocorrelation-function $\psi(t_1, t_2) := \langle v(t_1)v(t_2)\rangle$ and show that it becomes stationary $\psi(t_1, t_2) \equiv \psi(t_1 - t_2)$ if $m\Gamma = 4k_B T\gamma$.

Problem 32 *Master-Equation* **10 Punkte**

The probability $P_n(t)$ to find an N-state system in state n at time t shall be determined by the *Master Equation*

$$\frac{\partial P_n}{\partial t}(t) = \sum_m \left[w(n|m)P_m(t) - w(m|n)P_n(t)\right]$$

where $w(n|m)$ is the transition rate from state m to state n.

a) Show that the Master equation can be written more concisely as

$$\frac{\partial \boldsymbol{P}}{\partial t}(t) = \mathsf{W} \cdot \boldsymbol{P}(t)$$

with the probability vector $\boldsymbol{P}(t)$ and the transition matrix W.

b) Consider an asymmetric "random walk" through a cyclic 4-state system with

$$
\begin{aligned}
w(1|4) = w(4|3) = w(3|2) = w(2|1) &= \frac{3}{4} \\
w(1|2) = w(2|3) = w(3|4) = w(4|1) &= \frac{1}{4}
\end{aligned}
$$

and all other $w(n|m) = 0$. Determine the transition matrix W and find the stationary solution of the master equation.

c) Consider a system with discrete energy levels E_n and transition rates

$$
w(n|m) = \frac{e^{-\beta E_n}}{\cosh\big(\beta(E_n + E_m)\big)}
$$

Show that the stationary solution of the Master equation is the canonical distribution $P_n = \frac{1}{Z} e^{-\beta E_n}$.

Problem 33 *Hydrodynamic Modes* **12 Punkte**

The conservation equations for mass and momentum can be written in terms of the density field $\rho(\boldsymbol{r}, t)$ and the flow field $\boldsymbol{u}(\boldsymbol{r}, t)$ as

$$
\begin{aligned}
D_t \rho + \rho \nabla \cdot \boldsymbol{u} &= 0 \\
D_t \boldsymbol{u} + \frac{1}{\rho} \nabla \cdot \sigma &= 0
\end{aligned}
$$

where σ is the stress-tensor For an isotropic fluids, the second equation turns into the *Navier-Stokes-equation* if we make the ansatz

$$
\sigma_{ij} = p\delta_{ij} - \eta \left(\nabla_i u_j + \nabla_j u_i - \frac{2}{3} \delta_{ij} \nabla \cdot \boldsymbol{u} \right) - \zeta \delta_{ij} \nabla \cdot \boldsymbol{u}
$$

Here p denotes the pressure, and η, ζ the shear- and bulk-viscosity, respectively.

a) Consider a quiescent fluid at a given pressure p, a given mean density ρ_0 and vanishing mean flow $\boldsymbol{u} \equiv 0$. There shall be small fluctuations $\delta\rho(\boldsymbol{r}, t), \delta\boldsymbol{u}(\boldsymbol{r}, t)$ around the quiescent state. Consider the double Fourier transform in time and space of the fluctuation fields $\boldsymbol{a}(\boldsymbol{k}, \omega) := \big(\delta\rho(\boldsymbol{k}, \omega), u_\parallel(\boldsymbol{k}, \omega), \boldsymbol{u}_\perp(\boldsymbol{k}, \omega)\big)$ where $u_\parallel = \hat{\boldsymbol{k}} \cdot \delta\boldsymbol{u}$ is the longitudinal, and $\boldsymbol{u}_\perp = \delta\boldsymbol{u} - \hat{\boldsymbol{k}} u_\parallel$ the transverse component of the flow field. Linearize the hydrodynamic equations in $\boldsymbol{a}$

$$
i\omega \boldsymbol{a}(\boldsymbol{k}, \omega) = \mathsf{M}(k) \cdot \boldsymbol{a}(\boldsymbol{k}, \omega)
$$

to determine the hydrodynamic matrix $\mathsf{M}(k)$.

b) Calculate the eigen values of $\mathsf{M}(k)$ to second order in k and the corresponding eigen vectors to zeroth order in k. Interpret the terms in the eigen modes and discuss what physical properties they represent.