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WiSe 2016/17

12. Problem Set
Advanced Statistical Physics

Due Date: Thursday, January 26, 10am

Please indicate your name and the number of your group on the first page!

Problem 34 *Benford's law* **12 Punkte**

Benford's law encodes a curious observation: Take a large set of data (originally, logarthimic tables, nowadays famously the numbers from your tax declaration) and you record a histogram of the first digit $d \in \{1, 2, \ldots, 9\}$ you often find the following law for the probability distribution of said first digit

$$p(d) \approx p_B(d) := \log_{10}\left(1 + \frac{1}{d}\right).$$

In particular this predicts that the probability for the first digit being 1 is $\approx 30\%$ and that the digits are not uniformly distributed as one might have assumed. Benford's law cannot possibly hold for any arbitrary set of data. We will explore some of the sufficient conditions below

a) Show that the cumulative distribution function of Benford's law is given by $P_B(d) = \log_{10} d$

b) Convince yourself that

$$P(d) = \sum_{m=-\infty}^{\infty} \left[F(d \cdot 10^m) - F(10^m)\right]$$

in terms of the cumulative distribution $F(x)$ of the data. Benford's law will certainly *not* be valid for data with a clear scale (like, e.g., the height of people which clearly favours a narrow range of leading digits around the average height). Show that $P(d) = P_B(d)$ if we want $P(d)$ to be invariant under rescalings $x \to x/c$ of the data.

c) Using the transformed variable $y := \log_{10} x$ with $G(y) := F(10^y)$ show that Benford's law holds approximately if

$$1 \approx \sum_{m=-\infty}^{\infty} g(y + m)$$

Qualitatively, what are the conditions on $f(x)$ to fulfill this relation?

d) Find some data set (from the internet or from your practical courses) that you believe could follow Benford's law and test your hypothesis.

Problem 35 *Mori-Zwanzig & The Incoherent Scattering Function* **10 Punkte**

Derive the Mori-Zwanzig Equation of Motion for the incoherent scattering function $S^s(q, t)$ using a projector $\mathcal{P}$ that projects on the tagged-particle density $\rho^s_{\boldsymbol{q}} := e^{i\boldsymbol{q}\cdot\boldsymbol{r}_s}$ and the longitudinal part of the tagged-particle "current" $j^{sL}_{\boldsymbol{q}} := (\hat{\boldsymbol{q}} \cdot \boldsymbol{v}_s)e^{i\boldsymbol{q}\cdot\boldsymbol{r}_s}$. Here $\boldsymbol{r}_s, \boldsymbol{v}_s$ are the tagged particle position and velocity, respectively. Explicitly determine the elements of the frequency matrix.

Problem 36 *A Schematic Model for the Tagged-Particle Dynamics* **8 Punkte**

Lets assume the coherent scattering function $\phi(t)$ follows from the F_{12} model with a memory functional

$$m(t) = \nu_1 \phi(t) + \nu_2 \phi^2(t)$$

In problem 28 you showed that this model displays a glass transition from a fluid state where $\phi(t \to \infty) = 0$ to a glassy state with persistent correlations $\phi(t \to \infty) \equiv f > 0$. A minimal model to couple the tagged particle dynamics to the collective behaviour is given by making the following ansatz for the memory kernel in the schematic equation of motion for the incoherent scattering function $\phi^s(t)$ (problem 35)

$$m^s(t) = \lambda \phi(t) \phi^s(t)$$

with a coupling constant λ. Determine the *Lamb-Mössbauer* factors $f^s := \phi^s(t \to \infty)$ which follow from the same formula as the Debye-Waller factors f. Discuss the dependence of f^s on f and λ.