A classical long-time tail in a driven granular fluid

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Abstract. I derive a mode-coupling theory for the tagged particle velocity autocorrelation function, $\psi(t)$, in a fluid of randomly driven inelastic hard spheres far from equilibrium. With this, I confirm a conjecture from simulations that the velocity autocorrelation function decays algebraically, $\psi(t) \propto t^{-3/2}$, if momentum is conserved. I show that the slow decay is due to the coupling to transverse currents.

Keywords: granular matter, hydrodynamic fluctuations, mode coupling theory, kinetic theory of gases and liquids

Contents

1. Introduction

The algebraic, rather than exponential in time, decay of the (tagged particle) velocity autocorrelation function (VACF), $\psi(t) \propto t^{-\alpha}$, in simple fluids was quite a surprise when it was first discovered [1, 2]. It was finally explained by mode-coupling theories and attributed to vortex flows [3]. Long-time tails are expected even in high energy physics [4] now and have been reported recently also for fluids far from equilibrium [5]–[7]. In particular they are discussed for granular fluids [8]–[13].

Here, I show that the original mode-coupling argument [14]–[16] can be adapted to the stationary state of a randomly driven granular fluid. In particular, I explain the observation that $\alpha \approx 1.5$ in three space dimensions [13]. This is exactly the same exponent as for equilibrium fluids and stands in contrast to a number of unconventional exponents reported in the literature.

In a fluid in thermal equilibrium, long-time tails are a result of the coupling to the transverse current modes, $\boldsymbol{j}_{\boldsymbol{k}}^{\mathrm{T}}$, labelled by the wavevector $\boldsymbol{k}$. A number of approaches (see [3] and references therein, and [15]–[19]) confirmed the result $\psi(t \to \infty) \propto t^{-3/2}$. In a Lorentz gas, momentum is not conserved and it was argued [20] that this leads to a faster decay, $\psi(t \to \infty) \propto t^{-5/2}$. See [21] for why this behaviour may be hard to observe.

For a freely cooling granular gas, a long-time tail in the number of collisions, τ, is predicted, of the form $\psi(\tau \to \infty) \propto \tau^{-3/2}$ [9]. Here, the couplings to the longitudinal and transverse currents are both relevant. For shear-driven granular fluids, there are two competing proposals. Hayakawa and Otsuki [10] predict $\psi(t \to \infty) \propto (\dot{\gamma}t)^{-5/2}$, where $\dot{\gamma}$ is the shear rate, and Kumaran [12] predicts $\psi(t \to \infty) \propto (\dot{\gamma}t)^{-7/2}$ in the vorticity direction and a slightly faster decay, $\psi(t \to \infty) \propto (\dot{\gamma}t)^{-15/4}$, in the gradient and flow directions. The difference remains unresolved [22]. In both theories, the physical interpretation of the relevant collective modes is not obvious.

From the above discussion one can conclude that the existence of long-time tails seems to be rather universal even in fluids far from equilibrium. Two questions, however, have to be answered for every specific system: what is the mechanism that induces the slow decay and what is the value of the exponent α? In the following, I will address these two questions for the randomly driven granular fluid.

The paper is organized as follows. I start in section 2 by defining my model system. In section 3 I give the formally exact equation of motion for the VACF. This will be closed in section 4 with a mode-coupling approximation. In section 5 I discuss the results of the approximation, and in particular, the long-time tails. In the final section 6 I summarize my results and give some perspectives for future work.

2. The model

2.1. Inelastic hard spheres

The granular fluid is modelled as a monodisperse system of N smooth inelastic hard spheres of diameter d and mass $m = 1$ in a volume $V = L^3$. I consider the thermodynamic limit $N, V \to \infty$ such that the density $n = N/V$ remains finite. Dissipation is introduced through a constant coefficient of normal restitution $\varepsilon \in [0, 1]$ that augments the law of reflection [23],

$$\hat{\boldsymbol{r}}_{12} \cdot \boldsymbol{v}_{12}' = -\varepsilon \, \hat{\boldsymbol{r}}_{12} \cdot \boldsymbol{v}_{12}, \tag{1}$$

where $\boldsymbol{v}_{12} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ is the relative velocity and $\hat{\boldsymbol{r}}_{12}$ is the unit vector pointing from the centre of particle 2 to particle 1. The prime indicates post-collisional quantities.

2.2. The stochastic driving force

The driving force is implemented as an external random force,

$$\boldsymbol{v}_i'(t) = \boldsymbol{v}_i(t) + \sqrt{P_{\mathrm{D}}} \, \boldsymbol{\xi}_i(t), \tag{2}$$

where P_{D} is the driving power. The ξ_i^α, $\alpha = x, y, z$, are Gaussian random variables with zero mean and variance,

$$\left\langle \xi_i^\alpha(t) \xi_j^\beta(t') \right\rangle_\xi = [\delta_{ij} - \delta_{\pi(i),j}] \delta^{\alpha\beta} \delta(t - t'), \tag{3}$$

where $\pi(i)$ denotes the nearest neighbour of particle i. In effect, the two particles i and $\pi(i)$ are driven by forces of equal strength but opposite direction. Thereby, the external force does not destroy momentum conservation on macroscopic length scales [13, 24].

2.3. The granular fluid

Macroscopically, the fluid is fully characterized by the packing fraction, $\varphi = \pi n d^3/6$, the coefficient of restitution, ε, and the driving power, P_{D}. In the stationary state, the granular temperature $T = T(\varphi, \varepsilon, P_{\mathrm{D}}) = \frac{1}{3N} \sum_i \boldsymbol{v}_i^2$ is given by the balance between the driving power, P_{D}, and the energy loss through the inelastic collisions.

The collision frequency $\omega_c \propto \sqrt{T}$ is the only timescale of the system. Thus, changing the granular temperature only changes the timescale of the system. I use this freedom and set $T \equiv 1$ in the following.

3. A microscopic description

3.1. The phase space distribution

In contrast to the case for fluids in thermal equilibrium, no analytical expression for the stationary phase space distribution of driven granular fluids is known so far. Therefore, I have to make a few assumptions to evaluate the expectation values. First of all I assume that positions and velocities are uncorrelated, $\varrho(\Gamma) = \varrho_{\mathrm{r}}(\{\boldsymbol{r}_i\})\varrho_{\mathrm{v}}(\{\boldsymbol{v}_i\})$. Moreover, I assume that the velocity distribution factorizes into a product of one-particle distribution functions, $\varrho_{\mathrm{v}}(\{\boldsymbol{v}_i\}) = \prod_i \varrho_1(\boldsymbol{v}_i)$. All we need to know about $\varrho_1(\boldsymbol{v})$ is the situation for a few moments—namely, that it has a vanishing first moment, $\int \mathrm{d}^3v \, \boldsymbol{v}\varrho_1(\boldsymbol{v}) = 0$, a finite second moment, $\int \mathrm{d}^3v \, v^2\varrho_1(\boldsymbol{v}) = 3T < \infty$, and a finite third collisional moment, $\int \mathrm{d}^3v \, (\hat{\boldsymbol{r}} \cdot \boldsymbol{v})^3\Theta(-\hat{\boldsymbol{r}} \cdot \boldsymbol{v})\varrho_1(\boldsymbol{v}) < \infty$. The spatial distribution function, $\varrho_{\mathrm{r}}(\{\boldsymbol{r}_i\})$, enters the theory via a static correlation function, as will be discussed below.

Averages over pairs of observables define a scalar product, $\langle A|B\rangle := \langle A^*B\rangle := \int \mathrm{d}\Gamma \, \varrho(\Gamma)A^*(\Gamma)B(\Gamma)$, where A^* denotes the complex conjugate of A.

3.2. Observables

The VACF, $\psi(t) = \langle \boldsymbol{v}_s|\boldsymbol{v}_s(t)\rangle /3$, is defined in terms of the tagged particle velocity $\boldsymbol{v}_s$. The tagged particle position will be described by the density $\rho^s(\boldsymbol{r}, t) = \delta(\boldsymbol{r}_s - \boldsymbol{r}(t))$. The host fluid is characterized by the density and current fields

$$\rho(\boldsymbol{r}, t) = \frac{1}{N} \sum_i \delta(\boldsymbol{r} - \boldsymbol{r}_i(t)), \tag{4a}$$

$$\boldsymbol{j}(\boldsymbol{r}, t) = \frac{1}{N} \sum_i \boldsymbol{v}_i(t)\delta(\boldsymbol{r} - \boldsymbol{r}_i(t)). \tag{4b}$$

In particular, I will use the spatial Fourier transform of those fields, $\rho_{\boldsymbol{k}}^s(t), \rho_{\boldsymbol{k}}(t)$, and the longitudinal and transverse current fields $j_{\boldsymbol{k}}^L(t) = \hat{\boldsymbol{k}} \cdot \boldsymbol{j}_{\boldsymbol{k}}(t)$, and $\boldsymbol{j}_{\boldsymbol{k}}^T(t) = \boldsymbol{j}_{\boldsymbol{k}}(t) - \hat{\boldsymbol{k}}j_{\boldsymbol{k}}^L(t)$, respectively.

3.3. The dynamics

We have shown in [25] that the time evolution operator $U(t) = \exp(it\mathcal{L}_+)$ can be written in terms of an effective pseudo-Liouville operator $\mathcal{L}_+$ [26]. It is given as a sum of three parts, $\mathcal{L}_+ = \mathcal{L}_0 + \mathcal{T}_+ + \mathcal{L}_D^+$, which are in turn the free streaming operator $\mathcal{L}_0$, the collision operator $\mathcal{T}_+$, and the driving operator $\mathcal{L}_D^+$.

With the Mori projectors $\mathcal{P} = |\boldsymbol{v}_s\rangle\langle\boldsymbol{v}_s|/3$, $\mathcal{Q} = 1 - \mathcal{P}$, one can derive a formally exact equation of motion for the VACF:

$$\dot{\psi}(t) + \frac{1+\varepsilon}{3}\omega_{\mathrm{E}}\psi(t) + \omega_{\mathrm{E}}^2 \int_0^t \mathrm{d}\tau\, m(t-\tau)\psi(\tau) = 0, \tag{5}$$

where the local term $\langle\boldsymbol{v}_s|i\mathcal{L}_+\boldsymbol{v}_s\rangle/3 = -(1+\varepsilon)\omega_{\mathrm{E}}/3$ was determined in [13]. The memory kernel is formally given as

$$m(t) = \langle\boldsymbol{v}_s|\mathcal{L}_+\mathcal{Q}\tilde{U}(t)\mathcal{Q}\mathcal{L}_+\boldsymbol{v}_s\rangle/3\omega_{\mathrm{E}}^2 \tag{6}$$

and $\tilde{U}(t) = \exp(it\mathcal{Q}\mathcal{L}_+\mathcal{Q})$ is a modified propagator [27, 28, 25]. The Enskog collision frequency $\omega_{\mathrm{E}} = 24\varphi\chi/\sqrt{\pi}d$ is given in terms of the contact value of the pair correlation function at contact, χ [29].

4. Mode-coupling approximations

I consider three contributions to the memory kernel $m(t) \approx m_\rho(t) + m_{\mathrm{L}}(t) + m_{\mathrm{T}}(t)$ that are induced by the coupling of the tagged particle to the host fluid—namely, those to the collective density field ($m_\rho(t)$), and to the longitudinal and transverse current fields ($m_{\mathrm{L}}(t)$ and $m_{\mathrm{T}}(t)$, respectively).

The behaviour of the collective modes is characterized by their two-point correlation functions,

$$\phi(k,t) = N\langle\rho_{\boldsymbol{k}}|\rho_{\boldsymbol{k}}(t)\rangle/S_k, \tag{7a}$$

$$\phi_{\mathrm{L}}(k,t) = N\langle j_{\boldsymbol{k}}^{\mathrm{L}}|j_{\boldsymbol{k}}^{\mathrm{L}}(t)\rangle, \tag{7b}$$

$$\phi_{\mathrm{T}}^{\alpha\beta}(k,t) = N\langle j_{\boldsymbol{k}}^{\mathrm{T}\alpha}|j_{\boldsymbol{k}}^{\mathrm{T}\beta}(t)\rangle = \phi_{\mathrm{T}}(k,t)\delta^{\alpha\beta}, \tag{7c}$$

where $S_k = N\langle\rho_{\boldsymbol{k}}|\rho_{\boldsymbol{k}}\rangle$ is the static structure factor, and

$$\phi_{\mathrm{s}}(k,t) = \langle\rho_{\boldsymbol{k}}^{\mathrm{s}}|\rho_{\boldsymbol{k}}^{\mathrm{s}}(t)\rangle \tag{8}$$

is the incoherent scattering function.

In terms of these correlation functions, I replace the modified propagator

$$\tilde{U}(t) \approx N\sum_{\boldsymbol{k}}|\rho_{\boldsymbol{k}}\rho_{-\boldsymbol{k}}^{\mathrm{s}}\rangle\phi(k,t)\phi_{\mathrm{s}}(k,t)\langle\rho_{\boldsymbol{k}}\rho_{-\boldsymbol{k}}^{\mathrm{s}}|/S_k$$

$$+ N\sum_{\boldsymbol{k}}|j_{\boldsymbol{k}}^{L}\rho_{-\boldsymbol{k}}^{\mathrm{s}}\rangle\phi_{\mathrm{L}}(k,t)\phi_{\mathrm{s}}(k,t)\langle j_{\boldsymbol{k}}^{L}\rho_{-\boldsymbol{k}}^{\mathrm{s}}|$$

$$+ \frac{N}{2}\sum_{\boldsymbol{k}}|\boldsymbol{j}_{\boldsymbol{k}}^{\mathrm{T}}\rho_{-\boldsymbol{k}}^{\mathrm{s}}\rangle\phi_{\mathrm{T}}(k,t)\phi_{\mathrm{s}}(k,t)\langle j_{\boldsymbol{k}}^{\mathrm{T}}\rho_{-\boldsymbol{k}}^{\mathrm{s}}| \tag{9}$$

by a mode-coupling approximation. Similar approximations have been made in, e.g., [14, 15].

The coupling to the collective density field then reads

$$3\omega_{\mathrm{E}}^2 m_\rho(t) = \sum_{\boldsymbol{k}} \mathcal{V}_{\boldsymbol{k}}^\rho \mathcal{W}_{\boldsymbol{k}}^\rho \phi(k,t)\phi_{\mathrm{s}}(k,t), \tag{10}$$

where the vertices

$$\mathcal{V}_{\boldsymbol{k}}^\rho = \sqrt{N/S_k} \left\langle \boldsymbol{v}_s | \mathcal{L}_+ \mathcal{Q} \rho_{\boldsymbol{k}} \rho_{-\boldsymbol{k}}^{\mathrm{s}} \right\rangle = \boldsymbol{k}(S_k - 1)/\sqrt{NS_k}, \tag{11a}$$

$$\mathcal{W}_{\boldsymbol{k}}^\rho = \sqrt{N/S_k} \left\langle \rho_{\boldsymbol{k}} \rho_{-\boldsymbol{k}}^{\mathrm{s}} | \mathcal{Q} \mathcal{L}_+ \boldsymbol{v}_s \right\rangle = \frac{1+\varepsilon}{2} \boldsymbol{k}(S_k - 1)/\sqrt{NS_k} \tag{11b}$$

can be deduced from equations (46) and (47) in [25]. Explicitly, we find

$$\omega_{\mathrm{E}}^2 m_\rho(t) = \frac{2\pi^2}{9} \frac{1+\varepsilon}{2} \frac{d^3}{\varphi} \int_0^\infty \frac{\mathrm{d}k\,k^4}{(2\pi)^3} S_k (nc_k)^2 \phi(k,t)\phi_{\mathrm{s}}(k,t), \tag{12}$$

where $nc_k = 1 - 1/S_k$ is the direct correlation function [29]. This implies that $\omega_{\mathrm{E}}^2 m_\rho(t) \equiv m_0(t)$, where $m_0(t)$ is given in [30] as the memory kernel for the mean square displacement. In [30] we were concerned with the behaviour at high densities close to the glass transition and we used a mode-coupling approximation for the coherent scattering function, $\phi(k,t)$, itself. Here, I am interested in the regime of moderate densities, instead. Consequently, below I will use a hydrodynamic expression for the coherent scattering function (equation (20b)).

The coupling to the currents reads

$$3\omega_{\mathrm{E}}^2 m_{\mathrm{L}}(t) = \sum_{\boldsymbol{k}} \mathcal{V}_{\boldsymbol{k}}^L \mathcal{W}_{\boldsymbol{k}}^L \phi_{\mathrm{L}}(k,t)\phi_{\mathrm{s}}(k,t), \tag{13a}$$

$$3\omega_{\mathrm{E}}^2 m_{\mathrm{T}}(t) = \tfrac{1}{2} \sum_{\alpha,\beta} \sum_{\boldsymbol{k}} \mathcal{V}_{\boldsymbol{k}}^{\alpha\beta} \mathcal{W}_{\boldsymbol{k}}^{\alpha\beta} \phi_{\mathrm{T}}(k,t)\phi_{\mathrm{s}}(k,t), \tag{13b}$$

where the vertices

$$\mathcal{V}_{\boldsymbol{k}}^{\mathrm{L}} = \sqrt{N} \left\langle \boldsymbol{v}_s | \mathcal{L}_+ \mathcal{Q} j_{\boldsymbol{k}}^L \rho_{-\boldsymbol{k}}^{\mathrm{s}} \right\rangle, \tag{14a}$$

$$\mathcal{W}_{\boldsymbol{k}}^{\mathrm{L}} = \sqrt{N} \left\langle j_{\boldsymbol{k}}^L \rho_{-\boldsymbol{k}}^{\mathrm{s}} | \mathcal{Q} \mathcal{L}_+ \boldsymbol{v}_s \right\rangle, \tag{14b}$$

and

$$\mathcal{V}_{\boldsymbol{k}}^{\alpha\beta} = \sqrt{N} \left\langle v_s^\beta | \mathcal{L}_+ \mathcal{Q} j_{-\boldsymbol{k}}^{T\alpha} \rho_{\boldsymbol{k}}^{\mathrm{s}} \right\rangle, \tag{15a}$$

$$\mathcal{W}_{\boldsymbol{k}}^{\alpha\beta} = \sqrt{N} \left\langle j_{-\boldsymbol{k}}^{T\alpha} \rho_{\boldsymbol{k}}^{\mathrm{s}} | \mathcal{Q} \mathcal{L}_+ v_s^\beta \right\rangle \tag{15b}$$

are calculated in appendix A. While $\mathcal{V}_{\boldsymbol{k}}^\rho \neq \mathcal{W}_{\boldsymbol{k}}^\rho$ indicates the violation of time reversal invariance in the dissipative fluid, one finds

$$\mathcal{V}_{\boldsymbol{k}}^{\mathrm{L}} = \mathcal{W}_{\boldsymbol{k}}^{\mathrm{L}} = \mathrm{i}\frac{1+\varepsilon}{3} \hat{\boldsymbol{k}} \omega_{\mathrm{E}} U_{\mathrm{L}}(kd)/\sqrt{N}, \tag{16a}$$

$$\mathcal{V}_{\boldsymbol{k}}^{\alpha\beta} = \mathcal{W}_{\boldsymbol{k}}^{\alpha\beta} = \mathrm{i}\sqrt{2/3}\delta^{\alpha\beta}\frac{1+\varepsilon}{3} \omega_{\mathrm{E}} U_{\mathrm{T}}(kd)/\sqrt{N}, \tag{16b}$$

where $U_{\text{L}}(x) = 3j_0''(x)$ and $U_{\text{T}}(x) = \sqrt{6}j_0'(x)/x$ are effective potentials. Here, $j_0(x)$ is the zeroth-order spherical Bessel function [31] and the prime denotes the derivative with respect to the argument. Notably, the effective potentials are independent of density. The vertices are similar in form to those found in [14, 15].

For the memory kernels, we find[1]

$$m_{\text{L,T}}(t) = -\frac{8\pi^2}{81}\frac{(1+\varepsilon)^2}{4}\frac{d^3}{\varphi}\int_0^\infty \frac{\mathrm{d}k\,k^2}{(2\pi)^3}U_{\text{L,T}}^2(kd)\phi_{\text{L,T}}(k,t)\phi_{\text{s}}(k,t). \tag{17}$$

Given a static structure factor, S_k, and the dynamic correlator (7a)–(7c) and (8), the approximate memory kernel is fully determined by equations (12) and (17). All three contributions to the approximate memory kernel diverge in the short-time limit. Actually, the memory kernel should vanish for $t \to 0$. For elastic hard spheres, a number of proposals to that end have been made [14, 15, 17]. As I am only interested in the asymptotic behaviour, I will not further discuss this divergence.

5. Discussion

5.1. The long-time tail

The long-time asymptotics, $\psi(t \to \infty)$, are related to the limit $\lim_{s\to 0} s\hat{\psi}(s)$ in the Laplace domain[2]. For small s we have

$$\begin{aligned} s\hat{\psi}(s) &= s[-\mathrm{i}\omega_{\text{E}} + s - \omega_{\text{E}}^2\hat{m}(s)]^{-1}\\ &\simeq \mathrm{i}\frac{s}{\omega_{\text{E}}} + \frac{s^2}{\omega_E^2} - s\hat{m}(s), \end{aligned} \tag{18}$$

i.e., $\lim_{s\to 0} s\hat{\psi}(s) = -\lim_{s\to 0} s\hat{m}(s)$ or

$$\psi(t \to \infty) = -m(t \to \infty). \tag{19}$$

The long-time tails of the VACF are identical (up to the sign) to those of the associated memory kernel.

At moderate densities, a driven granular fluid is well described by Navier–Stokes order hydrodynamic equations [32, 33]. Consequently, I assume that the dynamic correlation functions take the following form:

$$\phi_{\text{s}}(k,t) = \mathrm{e}^{-Dk^2t}, \tag{20a}$$

where D is the diffusion coefficient,

$$\phi(k,t) = \cos(ckt)\mathrm{e}^{-\Gamma k^2t}, \qquad \phi_{\text{L}}(k,t) = \ddot{\phi}(k,t)/k^2, \tag{20b}$$

where c is the speed of sound and Γ the sound damping constant, and

$$\phi_{\text{T}}(k,t) = \mathrm{e}^{-\eta k^2t} \tag{20c}$$

with the shear viscosity η.

[1] The apparent divergence for $\varphi \to 0$ is spurious, as $\omega_{\text{E}} \sim \mathrm{O}(\varphi)$.
[2] I use the convention that $\hat{f}(s) = \mathsf{LT}[f](s) = \mathrm{i}\int_0^\infty f(t)\mathrm{e}^{-\mathrm{i}st}\,\mathrm{d}t$.

All the transport coefficients and the speed of sound are functions of the coefficient of restitution ε. For the diffusion coefficient, Fiege *et al* [13] found $D(\varepsilon) \propto 2/(1 + \varepsilon)$. According to van Noije *et al* [34] the sound damping constant is given as $\Gamma = \nu + D_\Gamma$ where ν is the kinematic viscosity and $D_\Gamma(\varepsilon) \propto 1/(1 - \varepsilon^2)$ is a term peculiar to inelastic fluids. The viscosities η and ν have a more complicated dependence on the degree of dissipation [35]. The speed of sound, c, is smaller in a fluid of inelastic compared to elastic hard spheres but only weakly depends on the value of the coefficient of restitution, ε [33, 34].

In the long-wavelength limit $k \to 0$ it holds that $S_k, c_k \to$ const. and $U_L^2(kd) \to 1$, $U_T^2(kd) \to 2/3$. In the long-time limit $t \to \infty$, we thus find

$$m_T(t \to \infty) \simeq -M_T[(D + \eta)t/d^2]^{-3/2}, \tag{21a}$$

$$m_L(t \to \infty) \simeq -M_L[(D + \Gamma)t/d^2]^{-3/2}\mathrm{e}^{-c^2t/4(D+\Gamma)}, \tag{21b}$$

$$m_\rho(t \to \infty) \simeq M_\rho[(D + \Gamma)t/d^2]^{-1/2}\mathrm{e}^{-c^2t/4(D+\Gamma)}. \tag{21c}$$

This is the central result of this contribution. The evaluation of $m_T(t \to \infty)$ is simply a moment of a Gaussian integral. The kinds of integrals that are necessary for the evaluation of $m_{\rho,L}(t \to \infty)$ are discussed in appendix B. Away from the glass transition, $c^2/4(D+\Gamma) \sim \mathrm{O}(\omega_E)$, i.e., the contributions $m_{\rho,L}(t)$ decay on a short timescale, $\propto \omega_E^{-1}$. The dominant asymptotic contribution is thus $m(t \to \infty) = m_T(t \to \infty)$.

The prefactors read explicitly

$$M_T = \frac{1}{486\sqrt{\pi}}\frac{(1 + \varepsilon)^2}{4\varphi}, \tag{22a}$$

$$M_L = \frac{1}{162\sqrt{\pi}}\frac{(1 + \varepsilon)^2}{4\varphi}\frac{c^2}{D + \Gamma}, \tag{22b}$$

$$M_\rho = \frac{1}{1152\sqrt{\pi}}\frac{1 + \varepsilon}{2\varphi}\frac{S_0(nc_0)^2}{\omega_E^2 d^2}\frac{c^4}{(D + \Gamma)^4}. \tag{22c}$$

Due to the nontrivial dependence of the viscosity, $\eta(\varphi, \varepsilon)$, and the sound damping $\Gamma(\varphi, \varepsilon)$ on the coefficient of restitution, ε, and on the density, φ, there is no simple trend of $m_{T,L,\rho}$ with ε. A reduction of the memory effects compared to the case for a fluid of elastic hard spheres, however, can be expected. The opposite trend was found for the long-time tail, $\propto t^{-3/2}$, of the autocorrelation function for the Eulerian flow field $\boldsymbol{u}(\boldsymbol{r}, t)$ in a suspension of rodlike active particles [5, 6]. It has been argued that in such a system the long-time tail could even be enhanced by a factor 1000 [5]. Interestingly, this enhancement is attributed to an increased noise temperature whereas here, an increase in the driving strength serves to partially suppress the correlations that support the long-time tail.

From equation (19), it follows that

$$\psi(t \to \infty) \simeq M_T[(D + \eta)t/d^2]^{-3/2} \propto t^{-3/2}. \tag{23}$$

With this result I have answered both questions from the introduction. We now know the value of the exponent α and which of the possible couplings is relevant.

At high densities, close to the granular glass transition [25], the viscosity, η, is expected to be large and the long-time tail will be strongly suppressed [36].

5.2. Signatures of non-equilibrium dynamics

At this point it is time to step back and ask whether it is possible to distinguish between a fluid of elastic hard spheres in thermal equilibrium and the specific model of a driven granular fluid discussed here and in [25, 30, 33], based on the tagged particle measurements of the VACF $\psi(t)$, or the incoherent scattering function $\phi_s(k,t)$. If we restrict ourselves to qualitative features and the theoretical results derived here and in [30], the answer is negative. All the dependence on the coefficient of restitution appears in such a way that it only leads to quantitative changes. The wavenumber dependence of the speed of sound is the only quantity that also changes qualitatively with the coefficient of restitution [25] but it does not, of course, qualitatively alter the tagged particle dynamics.

As discussed in [25], there is a reservation to this conclusion. The factorization of the phase space distribution function is a serious approximation. If it is avoided, and, e.g., the current correlation function $S_{\ell\ell}(k) = \langle j^L_{\bm{k}} | j^L_{\bm{k}} \rangle$ is treated as an irreducible quantity, additional terms will appear in the equations of motion that will also probably lead to new qualitative features. At present, the easiest way to detect the non-equilibrium stationary state is to look at $S_{\ell\ell}(k)$ directly [37].

6. Summary and perspectives

I discussed the coupling of the tagged particle velocity to the hydrodynamic modes of the host fluid in the framework of mode-coupling theory. Considering a randomly driven inelastic hard sphere fluid with local momentum conservation, I found that the VACF decays algebraically, $\psi(t \to \infty) \propto t^{-\alpha}$, with an exponent $\alpha = 3/2$. This supports observations from simulations [13]. The relevant process for the algebraic decay is, both for elastic and inelastic hard spheres, the coupling to the transverse currents. The couplings to the density and longitudinal currents have a finite lifetime. In general, the amplitude of the long-time tail will decrease with increasing dissipation or, equivalently, strength of driving.

The discussion of the VACF in a randomly driven granular fluid *without* momentum conservation will be left to future work. This could possibly help to settle the question about the nature of the long-time tails in the sheared granular fluid. Another step away from idealizations would be to relax the assumption of an infinite driving frequency and go to a more realistic finite frequency [38]. I hope that the explicit expressions for the amplitudes will aid experimenters to judge whether the long-time tails are observable in their set-ups.

Appendix A. Vertices

Here, I will detail the calculation of the vertices.

A.1. The longitudinal case

Due to the symmetry of the velocity distribution function, we have

$$
\begin{aligned}
\langle \boldsymbol{v}_s | \mathcal{L}_+ \mathcal{Q} j^{\mathrm{L}}_{-\boldsymbol{k}} \rho^{\mathrm{s}}_{\boldsymbol{k}} \rangle &= \langle \boldsymbol{v}_s | \rho^{\mathrm{s}}_{\boldsymbol{k}} \mathcal{T}_+ j^{\mathrm{L}}_{-\boldsymbol{k}} \rangle - \tfrac{1}{3} \langle \boldsymbol{v}_s | \mathcal{T}_+ \boldsymbol{v}_s \rangle \langle \boldsymbol{v}_s | j^{\mathrm{L}}_{-\boldsymbol{k}} \rho^{\mathrm{s}}_{\boldsymbol{k}} \rangle \\
&= \hat{\boldsymbol{k}} \langle j^{\mathrm{sL}}_{\boldsymbol{k}} | \mathcal{T}_+ j^{\mathrm{L}}_{\boldsymbol{k}} \rangle - \mathrm{i} \frac{1+\varepsilon}{3} \omega_{\mathrm{E}} \hat{\boldsymbol{k}} \langle j^{\mathrm{sL}}_{\boldsymbol{k}} | j^{\mathrm{L}}_{\boldsymbol{k}} \rangle
\end{aligned}
\tag{A.1}
$$

and

$$
\begin{aligned}
\langle j^{\mathrm{L}}_{-\boldsymbol{k}} \rho^{\mathrm{s}}_{\boldsymbol{k}} \mathcal{Q} | \mathcal{L}_+ \boldsymbol{v}_s \rangle &= \langle j^{\mathrm{L}}_{-\boldsymbol{k}} \rho^{\mathrm{s}}_{\boldsymbol{k}} | \mathcal{T}_+ \boldsymbol{v}_s \rangle - \tfrac{1}{3} \langle j^{\mathrm{L}}_{-\boldsymbol{k}} \rho^{\mathrm{s}}_{\boldsymbol{k}} | \boldsymbol{v}_s \rangle \langle \boldsymbol{v}_s | \mathcal{T}_+ \boldsymbol{v}_s \rangle \\
&= \hat{\boldsymbol{k}} \langle j^{\mathrm{L}}_{\boldsymbol{k}} | \mathcal{T}_+ j^{\mathrm{sL}}_{\boldsymbol{k}} \rangle - \mathrm{i} \frac{1+\varepsilon}{3} \omega_{\mathrm{E}} \hat{\boldsymbol{k}} \langle j^{\mathrm{L}}_{\boldsymbol{k}} | j^{\mathrm{sL}}_{\boldsymbol{k}} \rangle .
\end{aligned}
\tag{A.2}
$$

This shows that the left vertex and the right vertex are identical. With $\langle j^{\mathrm{L}}_{\boldsymbol{k}} | j^{\mathrm{sL}}_{\boldsymbol{k}} \rangle = 1/N$ and $\langle j^{\mathrm{L}}_{\boldsymbol{k}} | \mathcal{T}_+ j^{\mathrm{sL}}_{\boldsymbol{k}} \rangle = \nu_k / N$, where ν_k was determined in [25], equation (16a) follows.

A.2. The transverse case

Starting like in the longitudinal case, we find

$$
\langle \boldsymbol{j}^{\mathrm{T}}_{-\boldsymbol{k}} \rho^{\mathrm{s}}_{\boldsymbol{k}} \mathcal{Q} | \mathcal{L}_+ \boldsymbol{v}_s \rangle = \langle \boldsymbol{j}^{\mathrm{T}}_{\boldsymbol{k}} | \mathcal{T}_+ \boldsymbol{j}^{\mathrm{sT}}_{\boldsymbol{k}} \rangle - \mathrm{i} \frac{1+\varepsilon}{3} \omega_{\mathrm{E}} \langle \boldsymbol{j}^{\mathrm{T}}_{\boldsymbol{k}} | \boldsymbol{j}^{\mathrm{sT}}_{\boldsymbol{k}} \rangle .
\tag{A.3}
$$

The proof that the left and right vertices are identical is completely analogous to the discussion above.

We have $\langle \boldsymbol{j}^{\mathrm{T}}_{\boldsymbol{k}} | \boldsymbol{j}^{\mathrm{sT}}_{\boldsymbol{k}} \rangle = 2/N$ and $\langle \boldsymbol{j}^{\mathrm{T}}_{\boldsymbol{k}} | \mathcal{T}_+ \boldsymbol{j}^{\mathrm{sT}}_{\boldsymbol{k}} \rangle = \langle \boldsymbol{j}_{\boldsymbol{k}} | \mathcal{T}_+ \boldsymbol{j}^{\mathrm{s}}_{\boldsymbol{k}} \rangle - \langle j^{\mathrm{L}}_{\boldsymbol{k}} | \mathcal{T}_+ j^{\mathrm{sL}}_{\boldsymbol{k}} \rangle$. With $\boldsymbol{v} = (\boldsymbol{v}_1 - \boldsymbol{v}_s)/\sqrt{2}$ and $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_s$ we write

$$
\langle \boldsymbol{j}_{\boldsymbol{k}} | \mathcal{T}_+ \boldsymbol{j}^{\mathrm{s}}_{\boldsymbol{k}} \rangle = \mathrm{i} \frac{1+\varepsilon}{2} \sqrt{2} \left\langle (\hat{\boldsymbol{r}} \cdot \boldsymbol{v})^3 \Theta(-\hat{\boldsymbol{r}} \cdot \boldsymbol{v}) \delta(r-d) \left(\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} - 1 \right) \right\rangle
\tag{A.4}
$$

where $\sqrt{2} \langle (\hat{\boldsymbol{r}} \cdot \boldsymbol{v})^3 \Theta(-\hat{\boldsymbol{r}} \cdot \boldsymbol{v}) \rangle = -2/\sqrt{\pi}$ and

$$
\begin{aligned}
\langle \delta(r-d) \left(\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} - 1 \right) \rangle &= \frac{2\pi d^2 \chi}{V} \int_0^\pi \mathrm{d}\vartheta \, \sin\vartheta \left(\mathrm{e}^{-\mathrm{i}qd\cos\vartheta} - 1 \right) \\
&= -24 \frac{\varphi\chi}{dN} [1 - j_0(qd)],
\end{aligned}
\tag{A.5}
$$

i.e.,

$$
\langle \boldsymbol{j}_{\boldsymbol{k}} | \mathcal{T}_+ \boldsymbol{j}^{\mathrm{s}}_{\boldsymbol{k}} \rangle = 2\mathrm{i} \frac{1+\varepsilon}{2N} \omega_{\mathrm{E}} [1 - j_0(qd)].
\tag{A.6}
$$

Using suitable relations between spherical Bessel functions, equation (16b) follows.

Appendix B. Some integrals

All the integrals needed for $m_{\rho,\mathrm{L}}(t \to \infty)$ can be expressed as derivatives of

$$I(c,G;t) := \int_0^\infty \mathrm{d}k \, \cos(ckt)\mathrm{e}^{-Gk^2t} = \frac{1}{2}\sqrt{\frac{\pi}{Gt}}\exp(-c^2t/4G), \qquad (\mathrm{B.1})$$

where the second equality is given in [31]. Then we have

$$\int_0^\infty \mathrm{d}k \, k^2 \cos(ckt)\mathrm{e}^{-Gk^2t} = -\frac{1}{t}\frac{\partial I}{\partial G} = -\frac{1}{4}I(c,G;t)\frac{c^2t-2G}{G^2t}, \qquad (\mathrm{B.2})$$

$$\int_0^\infty \mathrm{d}k \, k^4 \cos(ckt)\mathrm{e}^{-Gk^2t} = \frac{1}{t^2}\frac{\partial^2 I}{\partial G^2} = \frac{1}{16}I(c,G;t)\frac{12G^2-12c^2Gt+c^4t^2}{G^4t^2}, \qquad (\mathrm{B.3})$$

and

$$\int_0^\infty \mathrm{d}k \, k^3 \sin(ckt)\mathrm{e}^{-Gk^2t} = \frac{1}{t^2}\frac{\partial^2 I}{\partial G \partial c} = \frac{1}{8}I(c,G;t)c\frac{c^2t-6G}{G^3t}. \qquad (\mathrm{B.4})$$

In particular,

$$2cG\frac{1}{t^2}\frac{\partial^2 I}{\partial G \partial c} + c^2\frac{1}{t}\frac{\partial I}{\partial G} = \frac{\sqrt{\pi}}{2}\frac{c^2}{G}(Gt)^{-3/2}\exp(-c^2t/4G). \qquad (\mathrm{B.5})$$

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